

ON THE DIMENSION DATUM PROBLEM AND THE LINEAR DEPENDENCE PROBLEM

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ABSTRACT. The dimension datum of a closed subgroup of a compact Lie group is the sequence of invariant dimensions of irreducible representations by restriction. In this article we classify closed connected subgroups with equal dimension data or linearly dependent dimension data. This classification should have applications to the isospectral geometry and automorphic form theory. We also study the equality/linear dependence of not necessarily connected subgroups of unitary group acting irreducibly on the natural representation.

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1. INTRODUCTION

Given a compact Lie group G , the dimension datum \mathcal{D}_H of a closed subgroup H of G is the map from \widehat{G} to \mathbb{Z} ,

$$\mathcal{D}_H : V \mapsto \dim V^H,$$

Key words and phrases. Dimension datum, Sato-Tate measure, root system, automorphism group, Weyl group, generating function.

where \widehat{G} is the set of equivalence classes of irreducible finite-dimensional complex linear representations of G and V^H is the subspace of H -invariant vectors in a complex linear representation V of G .

In number theory, dimension data arose from the determination of some monodromy groups (cf. [K]). In the theory of automorphic forms, Langlands ([La]) has suggested to use dimension data as a key ingredient in his program “Beyond Endoscopy”. The idea is to use the dimension datum to identify the conjectural subgroup ${}^\lambda H_\pi \subset {}^L \mathcal{G}$ associated to an automorphic representation π of $\mathcal{G}(\mathbb{A})$, where ${}^L \mathcal{G}$ is the L -group of \mathcal{G} in the form of [Bo, 2.4(2)]. In differential geometry, the dimension datum \mathcal{D}_H is related to the spectrum of the Laplace operator on the homogeneous Riemannian manifold G/H .

Let $H \subset G$ be compact Lie groups. The *dimension datum problem* asks the following question.

Question 1.1. *To what extent is H (up to G -conjugacy) determined by its dimension datum \mathcal{D}_H ?*

In [LP], Larsen and Pink considered the dimension data of connected semisimple subgroups. They showed that the subgroups are determined up to isomorphism by their dimension data (cf. [LP], Theorem 1), and not up to conjugacy in general (cf. [LP], Theorem 3). Moreover, they showed that the subgroups H are determined up to conjugacy if $G = \mathrm{SU}(n)$ and the inclusions $H \hookrightarrow \mathrm{SU}(n)$ give irreducible representations of H on \mathbb{C}^n (cf. [LP], Theorem 2).

In [La], Subsections 1.1 and 1.6, Langlands raised another question (cf. [AYY], Question 5.3) about dimension data. If this question has an affirmative answer, then it will facilitate a way of dealing with the dimension data of ${}^\lambda H_\pi$ using trace formulas. However, Langlands suspected (cf. [La], discussions following Equation (14)) that in general this question has an affirmative answer. As observed in [AYY], this question proposed by Langlands is equivalent to the following question that we shall call the *linear dependence problem*.

Question 1.2. *Given a list of finitely many closed subgroups H_1, \dots, H_n of G with $\mathcal{D}_{H_i} \neq \mathcal{D}_{H_j}$ for any $i \neq j$, are $\mathcal{D}_{H_1}, \dots, \mathcal{D}_{H_n}$ linearly independent?*

In [AYY], jointly with Jinpeng An and Jiu-Kang Yu, we have given counter-examples to show that the affirmative answer to Question 1.1 (or 1.2) is not always positive. In this paper, we classify closed connected subgroups with equal dimension data or linearly dependent dimension data. The method of this classification is as follows.

Given a compact Lie group G we choose a bi-invariant Riemannian metric m on it. For a closed connected torus T in G , we introduce root systems on T . Among the root systems on T , there is a maximal one, denoted by Ψ_T , which contains all other root systems on T . For each reduced root system Φ on T and a finite group W acting on $\mathfrak{t}_0 = \mathrm{Lie} T$ satisfying some condition (cf. Definition 3.12), we define a character $F_{\Phi, W}$. In particular, this definition applies to the finite group $\Gamma^\circ := N_G(T)/C_G(T)$. We define another root system Ψ'_T on T as the sub-root system of Ψ_T generated by the root systems of closed connected subgroups H of G with T a maximal torus of H . Moreover we show that the Weyl group of Ψ'_T is a sub-group of Γ° . By Propositions 3.15 and 3.16, we show that the dimension datum problem (or the linear dependence problem) reduces to comparing the characters $\{F_{\Phi, \Gamma^\circ} \mid \Phi \subset \Psi_T\}$ (or finding linear relations among them). We propose Questions 5.1 and 5.2. They are concerned with the characters associated to reduced sub-root systems of a given root system Ψ . In the next two sections we solve these two questions completely. We reduce both questions to the case where Ψ is an irreducible root system. In the case that Ψ is a classical irreducible root system, the classification of sub-root systems of Ψ is well-known. In [LP], Larsen and Pink defined an algebra isomorphism E taking characters of sub-root systems to polynomials (cf. Lemma 8.5 and Definition 8.6). In this case we

solve Questions 5.1 and 5.2 by getting all algebraic relations among polynomials in the image of E . The relations of some polynomials given in Propositions 8.11 and 9.3 are most important. In the case that Ψ is an exceptional irreducible root system, Oshima classified sub-root systems of Ψ (cf. [Os]). In [LP], the authors defined a weight $2\delta'_\Phi$ for each reduced sub-root system Φ of Ψ . We give the formulas of these weights in the Tables 2-9. We also define and calculate a generating function for each reduced sub-root system of Ψ (cf. Definition 9.12). With these, we are able to find and prove all linear relations among the characters of reduced sub-root systems of Ψ . Therefore we solve Questions 5.1 and 5.2 completely.

Given a connected closed torus T in G , the finite group Γ° plays an important role for the dimension data of closed connected subgroups H of G with T a maximal torus of H . In Proposition 3.9 we show that $\Gamma^\circ \supset W_{\Psi'_T}$, where Ψ'_T is the sub-root system of Ψ_T generated by root systems of closed connected subgroups H of G with T a maximal torus of H . On the other hand, in Proposition 10.6 we give a construction showing that the finite group Γ° could be arbitrary providing that it contains $W_{\Psi'_T}$. These supplements show that our solution to Questions 5.1 and 5.2 actually give all examples of closed connected subgroups with equal dimension data or linearly dependent dimension data.

In the last section, we study the equalities and linear relations among dimension data of closed subgroups of $U(n)$ acting irreducibly on \mathbb{C}^n . For nonconnected irreducible subgroups, we give interesting examples with equal or linearly dependent dimension data for $n = 16$ and $n = 12$ respectively. We also show that the dimension data of closed connected irreducible subgroups are actually linearly independent. This strengthens a theorem of Larsen and Pink. The proof is identical to theirs. The writing of this section is inspired by questions of Peter Sarnak.

The following theorem is a simple consequence of our classification of subgroups with equal dimension data. It follows from Proposition 3.16, Theorem 8.1, Theorem 8.13 and Theorem 8.14.

Theorem 1.1. *If two compact Lie groups H, H' have inclusions to a compact Lie group G with the same dimension data, then after replacing each ideal of $\text{Lie } H$ and $\text{Lie } H'$ isomorphic to $\mathfrak{u}(2n+1)$ ($n \geq 1$) by an ideal isomorphic to $\mathfrak{sp}(n) \oplus \mathfrak{so}(2n+2)$, the resulting Lie algebras are isomorphic.*

The classification of subgroups with linearly dependent dimension data is more complicated. The following theorem is a simple consequence of this classification. It follows from Propositions 9.9, 9.11 and 9.21. In the case of type B_2 , B_3 or G_2 , we also have a linear independence result. The proof requires some consideration of possible connected full rank subgroups.

Theorem 1.2. *Given $G = \text{SU}(n)$ or a compact connected Lie group isogenous to it, for any list $\{H_1, H_2, \dots, H_s\}$ of non-conjugate connected closed full rank subgroups of G , the dimension data $\mathcal{D}_{H_1}, \mathcal{D}_{H_2}, \dots, \mathcal{D}_{H_s}$ are linearly independent. Given a compact connected Lie group G with a simple Lie algebra \mathfrak{g}_0 , if \mathfrak{g}_0 is not of type A_n , B_2 , B_3 or G_2 , then there exist non-conjugate connected closed full rank subgroups H_1, H_2, \dots, H_s of G with linearly dependent dimension data.*

The organization of this paper is as follows. In Section 2, after recalling the definitions of root datum and root system, we define root system in a lattice. In Proposition 2.10 we show that a given lattice contains a unique maximal root system in it. In Section 3, given a compact Lie group G with a biinvariant Riemannian metric m , we define root systems on T and a maximal one Ψ_T among them. For each reduced root system Φ on T and a finite group W acting on T satisfying some condition, we define a character $F_{\Phi, W}$. Moreover, we discuss properties of these characters $\{F_{\Phi, W}\}$. Most importantly,

in Propositions 3.15 and 3.16, we reduce the dimension datum problem and the linear dependence problem to compare these characters and getting linear relations among them. In Section 4, we discuss the relations between several finite groups Γ° , Γ , $W_{\Psi'_T}$, W_{Ψ_T} and $\text{Aut}(\Psi_T)$. In particular we prove that $W_{\Psi'_T} \subset \Gamma^\circ$. Moreover, we discuss the connection between conjugacy relations of root systems of subgroups with regard to some of these finite groups and relations of the subgroups. In Section 5, we formulate two questions in terms of characters of reduced sub-root systems of a given root system. If the finite group Γ° contains W_{Ψ_T} , then these two questions are equivalent to the dimension datum problem and the linear dependence problem, respectively. In Section 6, we discuss the classification of reduced sub-root systems of a given irreducible root system. In Section 7, given an exceptional irreducible root system Ψ_0 , we give the formulas of the weights $\{2\delta'_\Phi \mid \Phi \subset \Psi_0\}$. In Section 8, given a root system Ψ , we classify reduced sub-root systems of Ψ with equal characters $F_{\Phi, \text{Aut}(\Psi)}$. This solves Question 5.1. In Section 9, given a root system Ψ , we classify reduced sub-root systems of Ψ with linearly dependent characters $F_{\Phi, W(\Psi)}$. This solves Question 5.2. In Section 10, given a root system Ψ' and a finite group W containing W_Ψ , we give examples of a compact connected simple Lie group G and a connected closed torus T in G satisfying that: $\dim T = \text{rank } \Psi'$, $\Psi' \subset \Psi_T$ and Ψ' is stable under Γ° , $\Gamma^\circ = W$ as groups acting on Ψ' and each reduced sub-root system Φ of Ψ' equals the root system of a closed connected subgrup H with T a maximal torus of H . In Section 11, we give interesting examples of irreducible subgroups of $U(n)$ with equal dimension data or linearly dependent dimension data, and show that connected irreducible subgroups of $U(n)$ actually have linearly independent dimension data.

Notation and conventions. Given a compact Lie group G ,

- (1) denote by G_0 the subgroup of connected component of G containing the identity element and $[G, G]$ the commutator subgroup.
- (2) Let $\mathfrak{g}_0 = \text{Lie } G$ be the Lie algebra of G .
- (3) Write G^\natural for the set of conjugacy classes in G ;
- (4) Denote by \widehat{G} the set of equivalence classes of irreducible finite-dimensional complex linear representations of G .
- (5) Let μ_G be the unique Haar measure on G with $\int_G 1 \mu_G = 1$.
- (6) Write V^G for the subspace of G -invariant vectors in a complex linear representation V of G .
- (7) Given a maximal torus S of G , let $W_G = N_G(S)/C_G(S)$ be the Weyl group of G . Then the quotient space S/W_G is a connected component of G^\natural . Moreover, in the case that G is connected, they are identical.
- (8) Given a closed subgroup H of G , denote by st_H the push-measure on G^\natural of μ_H under the composition map $H \hookrightarrow G \longrightarrow G^\natural$. It is called the *Sato-Tate measure* of the subgroup H .
- (9) For any complex linear representation ρ of G , $\mathcal{D}_H(\rho) = \int_H \chi_\rho(x) \mu_H = \text{st}_H(\chi_\rho^\natural)$, where χ_ρ^\natural is a continuous function on G^\natural induced from the character function χ_ρ of ρ . In this way, the dimension datum \mathcal{D}_H and the Sato-Tate measure st_H determine each other.

Given a compact Lie group G and a closed connected torus T , let $\Phi(G, T)$ be the set of non-zero weights of $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ as a complex linear representation of T . In Section 2, we explain that in some cases $\Phi(G, T)$ is a root system.

Given a lattice L ,

- (1) write $L_{\mathbb{Q}} = L \otimes_{\mathbb{Z}} \mathbb{Q}$ for the rational vector space generated by L .
- (2) Let $\mathbb{Q}[L]$ be the group ring of L over the field \mathbb{Q} .

- (3) For any element $\lambda \in L$, denote by $[\lambda]$ the corresponding element in $\mathbb{Q}[L]$. Then, for any $\lambda, \mu \in L$ and $n \in \mathbb{Z}$, $[\lambda][\mu] = [\lambda + \mu]$ and $[\lambda]^n = [n\lambda]$.

Given an abstract root system Φ (cf. Definition 2.5),

- (1) denote by $\mathbb{Z}\Phi$ the root lattice spanned by Φ and by $\mathbb{Q}\Phi = \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{Q}$ the rational vector space spanned by Φ .
 (2) Choosing a positive definite inner product $(\cdot, \cdot)_m$ on $\mathbb{Q}\Phi$ inducing the cusp product on Φ (which always exists and is unique if and only if Φ is irreducible), write

$$\Lambda_{\Phi} = \left\{ \lambda \in \mathbb{Q}\Phi \mid \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \in \mathbb{Z}, \forall \alpha \in \Phi \right\}$$

for the lattice of integral weights. One can show that Λ_{Φ} does not depend on the choice of the inner product m on $\mathbb{Q}\Phi$ if it induces the cusp product on Φ .

- (3) Let Φ° be the subset of short vectors in Φ : for any $\alpha \in \Phi$, $\alpha \in \Phi^{\circ}$ if and only if for any other $\beta \in \Phi$, $|\beta| \geq |\alpha|$ or $\langle \alpha, \beta \rangle = 0$. One can show that Φ° is a sub-root system of Φ .
 (4) If V is a rational (or real) vector space with a positive definite inner product $(\cdot, \cdot)_m$ containing Φ , let

$$\Lambda_{\Phi}(V) = \left\{ \lambda \in V \mid \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \in \mathbb{Z}, \forall \alpha \in \Phi \right\}.$$

Then $\Lambda_{\Phi}(V)$ is the direct sum of Λ_{Φ} and the linear subspace of vectors in V orthogonal to all vectors in Φ .

- (5) Given a subset X of Φ , we call the minimal sub-root system of Φ containing X (which exists and is unique) the sub-root system generated by X , and denote it by $\langle X \rangle$.

Write

$$T_k = \{ \text{diag}\{z_1, z_2, \dots, z_k\} \mid |z_1| = |z_2| = \dots = |z_k| = 1 \}.$$

We follow Bourbaki numbering to order the simple roots (cf. [Bou], Pages 265-300). Write ω_i for the i -th fundamental weight.

Acknowledgements. This article is a sequel of [AYY]. The author is grateful to Jinpeng An and Jiu-Kang Yu for the collaboration. He is also grateful to Brent Doran, Richard Pink, Gopal Prasad, Peter Sarnak and Jiu-Kang for helpful discussions and suggestions.

2. ROOT SYSTEM IN A LATTICE

In this section, after recalling the definitions of root datum and root system in a Euclidean vector space, we define root system in a lattice. Moreover, given a lattice, we show that it possesses a unique root system in it which contains all other root systems in it. Here lattice means a finite rank free abelian group with a positive definite inner product. Our discussion mostly follows [Bou], [Kn] and [Sp]. However there exist minor differences between our definitions and definitions in each of them. For example, our definition of abstract root system is different from that in [Kn], and we allow our root systems to be neither reduced nor semisimple, in contrast to all of the above references.

Defined as in [Sp], a root datum is a quadruple (X, R, X^*, R^*) together with some additional structure (duality and the root-coroot correspondence).

Definition 2.1. A root datum consists of a quadruple (X, R, X^*, R^*) , where

- (1) X and X^* are free abelian groups of finite rank together with a perfect pairing between them with values in \mathbb{Z} which we denote by (\cdot, \cdot) (in other words, each is identified with the dual lattice of the other).

- (2) R is a finite subset of X and R^* is a finite subset of X^* and there is a bijection from R onto R^* , denoted by $\alpha \mapsto \alpha^*$.
- (3) For each α , $(\alpha, \alpha^*) = 2$.
- (4) For each α , the map taking x to $x - (x, \alpha^*)\alpha$ induces an automorphism of the root datum (in other words it maps R to R and the induced action on X^* maps R^* to R^*).

Two root data (X_1, R_1, X_1^*, R_1^*) and (X_2, R_2, X_2^*, R_2^*) are called isomorphic if there exists a linear isomorphism $f : X_1 \rightarrow X_2$ such that $f(R_1) = f(R_2)$, $(f^*)^{-1}(R_1^*) = R_2^*$ and $f(\alpha)^* = ((f^*)^{-1})(\alpha^*)$ for any $\alpha \in R_1$. Here, $f^* : X_2^* \rightarrow X_1^*$ is the dual linear map of f and $(f^*)^{-1}$ is its inverse.

The elements of R are called *roots*, and the elements of R^* are called *coroots*. If R does not contain 2α for any α in R , then the root datum is called *reduced*. To a connected compact Lie group G with a maximal torus S , we can associate a reduced root datum $\text{RD}(G, S)$, whose isomorphism class $\text{RD}(G)$ depends only on G . Two connected compact Lie groups G, G' are isomorphic if and only if $\text{RD}(G)$ and $\text{RD}(G')$ are isomorphic.

A root system in a Euclidean vector space is a finite set with some additional structure.

Definition 2.2. Let V be a finite-dimensional Euclidean vector space, with an inner product m denoted by $(\cdot, \cdot)_m$. A root system in V is a finite set Φ of non-zero vectors (called roots) in V that satisfy the following conditions:

- (1) For any two roots α and β , the element $\beta - \frac{2(\beta, \alpha)_m}{(\alpha, \alpha)_m} \alpha \in \Phi$.
- (2) (**Integrality**) For any two roots α and β , the number $\frac{2(\beta, \alpha)_m}{(\alpha, \alpha)_m}$ is an integer.

Given a root system Φ in a Euclidean vector space V with an inner product m , we call $s_\alpha : V \rightarrow V$ defined by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \alpha, \quad \forall \lambda \in V$$

the reflection corresponding to the root α .

Moreover, a root system Φ in a Euclidean vector space V is called *semisimple* if the roots span V . It is called *reduced* if the only scalar multiples of a root $x \in \Phi$ that belong to Φ are x and $-x$.

To a connected compact semisimple Lie group G with a maximal torus S , we can associate a reduced semisimple root system $\text{R}(G, S)$, whose isomorphism class $\text{RD}(G)$ depends only on G . Two connected compact semisimple Lie groups G, G' having isomorphic universal covers if and only if $\text{R}(G)$ and $\text{R}(G')$ are isomorphic.

There are several different notions of lattice in the literature.

Definition 2.3. A lattice is a finite rank free abelian group with a positive definite inner product.

Now we define root systems in a lattice.

Definition 2.4. Let L be a lattice with a positive definite inner product m denoted by $(\cdot, \cdot)_m$. A root system in L is a finite set Φ of non-zero vectors (called roots) in L that satisfy the following conditions:

- (1) For any two roots α and β , the element $\beta - \frac{2(\beta, \alpha)_m}{(\alpha, \alpha)_m} \alpha \in \Phi$.
- (2) (**Strong integrality**) For any root α and any vector $\lambda \in L$, the number $\frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m}$ is an integer.

Given a root system Φ in a lattice L with an inner product m , we call $s_\alpha : L \rightarrow L$ defined by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \alpha, \quad \forall \lambda \in L$$

the reflection corresponding to the root α .

Moreover, a root system Φ in a lattice L is called *semisimple* if the roots span $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. It is called *reduced* if the only scalar multiples of a root $x \in \Phi$ that belong to Φ are x and $-x$.

If $\Phi \subset L$ is a root system in L , then it is a root system in $V = L \otimes_{\mathbb{Z}} \mathbb{R}$. Conversely, if $\Phi \subset L$ is a root system in V , it is not necessarily a root system in L . For example, let $L = \mathbb{Z}e_1$ be a rank 1 lattice with $(e_1, e_1) > 0$ and $V = \mathbb{R}e_1$. Then for any $k \in \mathbb{Z}_{>0}$, $\{\pm ke_1\}$ is a root system in V . However $\{\pm ke_1\}$ is a root system in L only if $k = 1$ or 2 . This example indicates that strong integrality is strictly stronger than integrality.

Definition 2.5. Given a root system Φ in a Euclidean vector space or a lattice with a positive definite inner product m , we call

$$\langle y, x \rangle = \frac{2(x, y)_m}{(x, x)_m}$$

the *cuspidal product* on Φ . Overlooking the Euclidean space or the lattice containing Φ , we call Φ with the cuspidal product on it an *abstract root system*.

Different with the above definition, an abstract root system as defined in the book [Kn] is simply a semisimple root system in a Euclidean vector space in our sense.

Proposition 2.6. Given an abstract root system Φ , for two roots $\alpha \in \Phi$ and $\beta \in \Phi$, let $k_1 = \max\{k \in \mathbb{Z} \mid \beta + k\alpha \in \Phi\}$ and $k_2 = \max\{k \in \mathbb{Z} \mid \beta - k\alpha \in \Phi\}$. Then $\langle \beta, \alpha \rangle = k_2 - k_1$.

Proof. Applying the reflection s_α , we get $\langle \beta + k_1\alpha, \alpha \rangle + \langle \beta - k_2\alpha, \alpha \rangle = 0$. Thus $\langle \beta, \alpha \rangle = k_2 - k_1$ follows from $\langle \alpha, \alpha \rangle = 2$. \square

Definition 2.7. An abstract root system Φ with a cuspidal product $\langle \cdot, \cdot \rangle$ is called *irreducible* if there exist no non-trivial disjoint unions $\Phi = \Phi_1 \cup \Phi_2$ such that $\langle \alpha_1, \alpha_2 \rangle = 0$ for any $\alpha_1 \in \Phi_1$ and $\alpha_2 \in \Phi_2$. A root system Φ in a Euclidean vector space or in a lattice is called *irreducible* if it is semisimple and irreducible as an abstract root system.

Definition 2.8. Given a root system Φ in a Euclidean vector space V with a positive definite inner product m , denote by $\text{Aut}(\Phi, m)$ the group of linear isomorphisms of V stabilizing Φ and preserving the inner product m , and by W_Φ the group of linear isomorphisms of V generated by $\{s_\alpha \mid \alpha \in \Phi\}$.

Given a root system Φ in a lattice L with a positive definite inner product m , write $\text{Aut}(\Phi, m)$ for the group of linear isomorphisms of L stabilizing Φ and preserving the inner product m , and W_Φ for the group of linear isomorphisms of L generated by $\{s_\alpha \mid \alpha \in \Phi\}$.

Given an abstract root system Φ with a cuspidal product $\langle \cdot, \cdot \rangle$, let $\text{Aut}(\Phi)$ be the group of permutations on Φ preserving its cuspidal product, and by W_Φ the group of permutations on Φ generated by $\{s_\alpha \mid \alpha \in \Phi\}$.

By definitions we have

$$W_\Phi \subset \text{Aut}(\Phi, m),$$

and the restriction on Φ gives a homomorphism

$$\pi : \text{Aut}(\Phi, m) \rightarrow \text{Aut}(\Phi).$$

Given a root system Φ in a Euclidean vector space V , we have the following characterization of lattices L in V such that Φ is also a root system in L .

Proposition 2.9. *Given a Euclidean vector space V with a positive definite inner product m denoted by (\cdot, \cdot) , let Φ be a root system in V . Then for any lattice $L \subset V$, Φ is a root system in L if and only if $\mathbb{Z}\Phi \subset L \subset \Lambda_\Phi(V)$.*

Proof. Suppose Φ is a root system in L . Since $\Phi \subset L$, $\mathbb{Z}\Phi \subset L$. On the other hand, by the condition strong integrality in Definition 2.4, $L \subset \Lambda_\Phi(V)$. Hence $\mathbb{Z}\Phi \subset L \subset \Lambda_\Phi(V)$.

Suppose $\mathbb{Z}\Phi \subset L \subset \Lambda_\Phi(V)$. Then $\Phi \subset L$ and strong integrality holds. On the other hand, since Φ is a root system in V , it is stable under the reflections $\{s_x \mid x \in \Phi\}$. Hence Φ is a root system in L . \square

Recall that a root system Φ in a Euclidean space (or in a lattice) with an inner product m is called *simply laced* if it is irreducible and $(\alpha, \alpha)_m = (\beta, \beta)_m$ for any $\alpha, \beta \in \Phi$. Similarly, an abstract root system Φ is called simply laced if it is irreducible and its cusp product takes values in the set $\{0, 1, -1\}$.

An embedding (resp. isomorphism) of root systems in lattices

$$f : (L, \Phi, m) \longrightarrow (L', \Phi', m')$$

is a \mathbb{Z} -linear bijection $f : L \rightarrow L'$ which is an isometry with respect to (m, m') and satisfying that $f(\Phi) \subset \Phi'$ (resp. $f(\Phi) = \Phi'$). If $L = L'$, $m = m'$ and f is the identity, we simply say that Φ is a sub-root system of Φ' . Similarly, we define embeddings and sub-root systems for root systems in Euclidean vector spaces.

We remark that in the literature, root systems in a Euclidean vector space are often required to be semisimple and/or reduced (e.g. [Hum] and [Kn]). We require neither. And it is essential for us to include the inner product in our definition of root systems in a Euclidean vector space or in a lattice.

For simplicity, we abbreviate as root system for a root system in a Euclidean vector space, a root system in a lattice or an abstract root system simply when it is clear from context which is intended. A root system in a lattice (L, Φ, m) (or a root system in a Euclidean vector space (V, Φ, m)) will be denoted simply by Φ in the case that the lattice L and the inner product m (or the Euclidean vector space V and the inner product m) are clear from the context.

Proposition 2.10. *Given a lattice L with a positive definite inner product m denoted by $(\cdot, \cdot)_m$, there exists a root system Ψ_L in L containing all other root systems in L . The root system Ψ_L is uniquely determined by this characterization.*

Proof. Define

$$\Psi_L = \left\{ 0 \neq \alpha \in L \mid \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \in \mathbb{Z}, \forall \lambda \in \Lambda \right\}.$$

We show that Ψ_L satisfies the desired conclusion.

First we show that Ψ_L is a root system in L . Obviously Ψ_L satisfies the condition (2) in Definition 2.4. For any $\alpha, \beta \in \Psi$ and any $\lambda \in L$,

$$\begin{aligned} & \frac{2(\lambda, s_\alpha(\beta))_m}{(s_\alpha(\beta), s_\alpha(\beta))_m} \\ = & \frac{2(s_\alpha(\lambda), \beta)_m}{(\beta, \beta)_m} \\ = & \frac{2(\lambda - \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \alpha, \beta)_m}{(\beta, \beta)_m} \\ = & \frac{2(\lambda, \beta)_m}{(\beta, \beta)_m} - \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \frac{2(\alpha, \beta)_m}{(\beta, \beta)_m}. \end{aligned}$$

Since $\alpha, \beta \in \Psi_L$ and $\alpha, \lambda \in L$, the number in the last line is an integer. Hence the vector $s_\alpha(\beta)$ is contained in Ψ_L . This proves the condition (1). Therefore Ψ_L is a root system in L .

By the condition strong integrality in Definition 2.4, Ψ_L contains all other root systems in L . This indicates that Ψ_L is unique with this characterization. \square

From Proposition 2.10 and its proof, we see that the condition strong integrality really matters. In the following proposition, we give some examples of root systems in lattices. These root systems will be used in the proof of Proposition 10.1.

Proposition 2.11. *Let G be a connected compact Lie group with a biinvariant Riemannian metric m , T a connected closed torus in G , $L = \text{Hom}(T, \text{U}(1))$ the weight lattice, and m the induced positive definite inner product on L . Then the set $\Phi(G, T)$ of non-zero T -weights in $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a root system in L in the case that*

- (1) T is a maximal torus of G ,
- (2) there exists an involutive automorphism θ of \mathfrak{g}_0 and $\text{Lie } T$ is a maximal abelian subspace of $\mathfrak{g}_0^{-\theta} = \{X \in \mathfrak{g}_0 | \theta(X) = -X\}$, or
- (3) $\mathfrak{g}_0 = \mathfrak{so}(8)$ and $\text{Lie } T$ is conjugate to a Cartan subalgebra of a (unique up to conjugacy) subalgebra of \mathfrak{g}_0 isomorphic to \mathfrak{g}_2 .

Proof. We prove (1) first. It is well-known that $\Phi(G, T)$ is a root system in $L \otimes_{\mathbb{Z}} \mathbb{R} = \text{Hom}_{\mathbb{R}}(\text{Lie } T, i\mathbb{R})$. Thus it satisfies the condition (1) in Definition 2.4. We need to show that it satisfies the condition (2). Write $r = \dim T = \text{rank } G$. For each $\alpha \in \Phi(G, T)$, there exists a homomorphism (cf. [Kn], Pages 143-149)

$$f_\alpha : \text{SU}(2) \times \text{U}(1)^{r-1} \longrightarrow G$$

such that $f_\alpha(\text{U}(1)^{r-1}) = (\ker \alpha)_0$, $f_\alpha(T' \times \text{U}(1)^{r-1}) = T$, and the complexified Lie algebra of $f_\alpha(\text{SU}(2))$ is the \mathfrak{sl}_2 subalgebra corresponding to the root α . Here T' is the subgroup of $\text{SU}(2)$ of diagonal matrices. We have $T' \cong \text{U}(1)$ and this gives a linear character α_0 of T' . We can normalize α_0 such that $\alpha \circ (f_\alpha|_{T'})$ equals $2\alpha_0$. Let

$$W = N_G(T)/C_G(T),$$

$$n_\alpha = f_\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

and

$$s_\alpha = \text{Ad}(n_\alpha) \in W.$$

For any $\lambda \in L$, let $k\alpha_0 = \lambda \circ (f_\alpha|_{T'})$. Then $k \in \mathbb{Z}$. We prove that $(2\lambda - k\alpha, \alpha)_m = 0$. This is equivalent to $s_\alpha(2\lambda - k\alpha) = 2\lambda - k\alpha$, and also equivalent to

$$(1) \quad (2\lambda - k\alpha)(s_\alpha(H)) = (2\lambda - k\alpha)(H)$$

for any $H \in \text{Lie } T$. Write H as $H = H_1 + H_2$ where $H_1 \in \text{Lie}(f_\alpha(T'))$ and $H_2 \in \text{Lie}(\ker \alpha)$. Then we have $s_\alpha(H_1) = -H_1$ and $s_\alpha(H_2) = H_2$. By this Equation (1) is equivalent to $(2\lambda - k\alpha)(H_1) = 0$. This follows from $k\alpha_0 = \lambda \circ (f_\alpha|_{T'})$ and $\alpha \circ (f_\alpha|_{T'}) = 2\alpha_0$. Thus

$$\frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} = \frac{2(\frac{k}{2}\alpha, \alpha)_m}{(\alpha, \alpha)_m} = k \in \mathbb{Z}.$$

Hence $\Phi(G, T)$ satisfies the condition (2) in Definition 2.4.

The proof for (2) is similar as the proof for (1). In [Kn], Pages 379-380, it is proven that $\Phi(G, T)$ is a root system in $L \otimes_{\mathbb{Z}} \mathbb{R} = \text{Hom}_{\mathbb{R}}(\text{Lie } T, i\mathbb{R})$. Thus it satisfies the condition (1) in Definition 2.4. Similar as in the above proof, there is a reflection $s_\alpha \in N_G(T)/C_G(T)$. Using it we are able to show that $\Phi(G, T)$ satisfies the condition (2) in Definition 2.4.

For (3), let H denote a closed subgroup of H isomorphic to G_2 and with T a maximal torus of H . Then we have $\Phi(G, T) = \Phi(H, T)$. In this way (3) follows from (1). \square

In the theory of integral lattice, there is a notion of root system of a lattice, which is different with our discussion here.

Remark 2.12. *Given a lattice L with a positive definite inner product m denoted by (\cdot, \cdot) , it is called an integral lattice if $(\lambda_1, \lambda_2) \in \mathbb{Z}$ for any $\lambda_1, \lambda_2 \in L$. In this case the set $X = \{\lambda \in L \mid (\lambda, \lambda) = 1 \text{ or } 2\}$ is called the root system of L . In general X is a proper subset of Ψ_L .*

Remark 2.13. *Root systems (in Euclidean vector spaces) classify compact connected Lie groups up to isogeny, however root data classify them up to isomorphism. Hence the difference between root datum and root system in a Euclidean vector space is clear. In another direction, we think that root datum and root system in a lattice should be equivalent objects. That is to say, there should exist a construction from a root datum to a root system in a lattice, and vice versa.*

3. THE LARSEN-PINK METHOD

In this section, given a compact Lie group G with a biinvariant Riemannian metric m and a closed connected torus T in G , we define root systems on T and get two specific root systems Ψ_T and Ψ'_T . Moreover, we define a character $F_{\Phi, W}$ for each root system Φ on T and a finite group W acting on the Lie algebra of T and satisfying some condition (cf. Definition 3.12). We study properties of the character $F_{\Phi, W}$ and their relation with dimension data. In this way, we reduce the dimension datum problem and the linear dependence problem to comparing these characters and getting linear relations among them.

Let G be a compact Lie group with a biinvariant Riemannian metric m . For a closed connected torus T contained in G , let

$$\Lambda_T = \text{Hom}(T, U(1))$$

be the integral weight lattice of T and $\Lambda_{\mathbb{Q}, T} = \Lambda_T \otimes_{\mathbb{Z}} \mathbb{Q}$ the \mathbb{Q} -weight space of T . By restriction the Riemannian metric m induces an inner product on the Lie algebra $\mathfrak{t}_0 = \text{Lie } T$ of T . Denote it by m as well. Write $(\mathfrak{t}_0)^* = \text{Hom}_{\mathbb{R}}(\mathfrak{t}_0, \mathbb{R})$ and $(\mathfrak{t}_0)_{\mathbb{C}}^* = \text{Hom}_{\mathbb{R}}(\mathfrak{t}_0, \mathbb{C})$. Then $(\mathfrak{t}_0)^* \subset (\mathfrak{t}_0)_{\mathbb{C}}^*$ and $(\mathfrak{t}_0)_{\mathbb{C}}^* = (\mathfrak{t}_0)^* \oplus i(\mathfrak{t}_0)^*$. Since m is positive definite, in particular non-degenerate, the induced linear map

$$p : \mathfrak{t}_0 \longrightarrow (\mathfrak{t}_0)^* = \text{Hom}_{\mathbb{R}}(\mathfrak{t}_0, \mathbb{R})$$

is a linear isomorphism. For any $\lambda, \mu \in (\mathfrak{t}_0)^*$, let

$$(\lambda, \mu)_m = -(p^{-1}\lambda, p^{-1}\mu)_m.$$

Then $(\cdot, \cdot)_m$ is a negative definite inner product on $(\mathfrak{t}_0)^*$. For any $\lambda_1, \lambda_2, \mu_1, \mu_2 \in (\mathfrak{t}_0)^*$, let

$$(\lambda_1 + i\lambda_2, \mu_1 + i\mu_2)_m = ((\lambda_1, \mu_1)_m - (\lambda_2, \mu_2)_m) + i((\lambda_1, \mu_2)_m + (\lambda_2, \mu_1)_m).$$

Then $(\cdot, \cdot)_m$ is a non-degenerate symmetric bilinear form on $(\mathfrak{t}_0)_{\mathbb{C}}^*$. It is positive definite on $i(\mathfrak{t}_0)^*$ by restriction. Since $\Lambda \subset \Lambda_{\mathbb{Q}} \subset i\mathfrak{t}_0^*$, they inherit this inner product.

Definition 3.1. *Let Γ be the group of automorphisms of T preserving the Riemannian metric m on it, and let*

$$\Gamma^{\circ} = N_G(T)/C_G(T).$$

By this definition we have $\Gamma^\circ \subset \Gamma$. The group Γ has an equivalent definition as the group of automorphisms of the lattice Λ_T preversing the inner product m on Λ_T . Choose a maximal torus S of G containing T and let $W_G = N_G(S)/C_G(S)$ be the Weyl group of G . Then Γ° has an equivalent definition as the image of $\text{Stab}_{W_G}(T)$ in $\text{Aut}(T)$. Given a non-zero element $\alpha \in \Lambda_T$, define $s_\alpha : i\mathfrak{t}_0^* \rightarrow i\mathfrak{t}_0^*$ by

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \alpha, \quad \forall \lambda \in i\mathfrak{t}_0^*.$$

Then s_α is a reflection on $i\mathfrak{t}_0^*$.

Definition 3.2. A subset $\Phi \subset \Lambda_T$ is called a root system on T if for any $\alpha \in \Phi$ and $\lambda \in \Lambda_T$,

$$s_\alpha(\Phi) = \Phi$$

and

$$\frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \in \mathbb{Z}.$$

In other words, a subset $\Phi \subset \Lambda_T$ is a root system on T if it is a root system in Λ_T .

Definition 3.3. Let

$$\Psi_T = \left\{ 0 \neq \alpha \in \Lambda_T \mid \frac{2(\lambda, \alpha)_m}{(\alpha, \alpha)_m} \in \mathbb{Z}, \quad \forall \lambda \in \Lambda_T \right\}.$$

By Proposition 2.10, Ψ_T is the unique maximal root system in the lattice Λ_T . By the definition Ψ_T is stable under the action of $\Gamma = \text{Aut}(\Lambda_T, m)$. Thus we have

$$\Gamma = \text{Aut}(\Psi_T, m),$$

where $\text{Aut}(\Psi_T, m)$, the automorphism of the root system Ψ_T in the lattice Λ_T , is defined in Definition 2.8.

Proposition 3.4. For any closed connected subgroup H of G with T a maximal torus of H , the root system of H , $\Phi(H) = \Phi(H, T)$ is a root system on T . Moreover, it is a reduced root system. Defined as above, Ψ_T is a root system on T and it contains all root systems on T . In particular $\Phi(H, T) \subset \Psi_T$ for any closed subgroup H of G with T a maximal torus of H .

Proof. The fact that subset $\Phi(H, T)$ is a root system on T is proven in Proposition 2.11. The fact that the subset Ψ_T is a root system on T and contains all other root systems on T is proved in Proposition 2.10. \square

In general the rank of Ψ_T can be any integer between 0 and $\dim T$. It happens that $\text{rank } \Psi_T = \dim T$ if and only if there exists a root system on T of rank equal to $\dim T$. In particular if there exists a semisimple closed subgroup H of G with T a maximal torus of H , then $\text{rank } \Psi_T$ is equal to $\dim T$. We choose and fix an ordering on Λ_T so that we get a positive system $(\Psi_T)^+$.

Remark 3.5. In [LP], Larsen-Pink defined a root system Ψ from a dimension datum. In general this Ψ is a root system on T and is strictly contained in our Ψ_T .

The root system Ψ_T depends on the torus T and a biinvariant Riemannian metric $m|_T$. If we replace G by a group isogenous of it, then the lattice Λ_T becomes another lattice isogeneous to it. The root system Ψ_T will change probably after this modification. On the other hand, Ψ_T is also sensitive with the biinvariant Riemannian metric $m|_T$. In a different direction, we can define another root system Ψ'_T which depends on T and the isogeny class of G . However, if we do not know G and T explicitly, it is hard to determine Ψ'_T .

Definition 3.6. Define Ψ'_T as the sub-root system of Ψ_T generated by $\{\Phi(H, T)\}$, where H runs through all closed connected subgroups H of G with T a maximal torus of H .

It is clear that Ψ'_T is stable under Γ° . However it is not stable under W_{Ψ_T} , Γ or $\text{Aut}(\Psi_T)$ in general.

Remark 3.7. With Ψ_T , we are able to consider all root systems

$$\{\Phi(H, T) \mid T \text{ is a maximal torus of } H\}$$

together by viewing them as sub-root systems of Ψ_T . In the case that G is a connected simple Lie group and T is a maximal torus of G , let $\Psi_0 = \Phi(G, T)$ be the root system of G . Then $\Psi_0 \subset \Psi_T$. If Ψ_0 is simply laced, then for any sub-root system Φ of Ψ_0 there exists a closed subgroup H of G with T a maximal torus of H such that $\Phi(H, T) = \Phi$. If Ψ_0 is not simply laced, then not every sub-root system Φ of Ψ_0 is of the form $\Phi(H, T) = \Phi$ where H is a closed subgroup of G with T a maximal torus of H . In the case that Ψ_0 is of type A_n ($n \geq 4$), E_6 , E_7 , E_8 , F_4 or G_2 , it must be that $\Psi_T = \Psi_0$. However, if Ψ_0 is of type A_2 , B_n , C_n or D_n ($n \geq 3$), then Ψ_T may be strictly larger than Ψ_0 . Precisely to say, in the case that $\Psi_0 = A_2$, it is possible that $\Psi_T = G_2$. In the case that $\Psi_0 = B_n$ or C_n , it is possible that $\Psi_T = BC_n$. In the case that $\Psi_0 = D_n$, it is possible that $\Psi_T = B_n$, C_n or BC_n .

Lemma 3.8. Given a root system Ψ and its Weyl group W_Ψ , let X be a non-empty subset of Ψ and W be the subgroup of W_Ψ generated by reflections corresponding to elements in X . If there exist no proper sub-root systems of Ψ containing X , then $W = W_\Psi$.

Proof. Let $X' = \{\alpha \in \Psi \mid s_\alpha \in W\}$. For any two roots α, β contained in X' , since

$$s_{-\alpha} = s_\alpha \in W$$

and

$$s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha \in W,$$

we get $-\alpha \in X'$ and $s_\alpha(\beta) \in X'$. That means, X' is a sub-root system of Ψ . On the other hand, we have $X \subset X'$. By the condition that there exist no proper sub-root systems of Ψ containing X , we get $X' = \Psi$. Therefore $W = W_\Psi$. \square

Proposition 3.9. Given a compact Lie group G and a biinvariant Riemannian metric m on G , for any closed connected torus T in G , we have $W_{\Psi'_T} \subset \Gamma^\circ$

Proof. Recall that Ψ'_T is the sub-root system of Ψ_T generated by $\{\Phi(H, T) \subset \Psi_T\}$, where H run over closed connected subgroups of G with T a maximal torus of H . Let X be the subset of Ψ'_T defined as: $\alpha \in X$ if and only if there exists a closed connected subgroup H of G with T a maximal torus of H and the root system of H being $\{\pm\alpha\}$. Let W be the subgroup of $W_{\Psi'_T}$ generated by reflections corresponding to elements in X .

We show that $W \subset \Gamma^\circ$ and there exist no proper sub-root systems of Ψ'_T containing X . For any $\alpha \in X$, there exists a closed connected subgroup H of G with T a maximal torus of H and the root system of H being $\{\pm\alpha\}$. Thus there exists a finite surjection

$$p : \text{SU}(2) \times \text{U}(1)^{r-1} \longrightarrow H.$$

Let

$$n_\alpha = p\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right),$$

where $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SU}(2)$ and $1 \in \text{U}(1)^{r-1}$. Then $n_\alpha \in N_G(T)$ and $\text{Ad}(n_\alpha)|_T = s_\alpha$. Hence $s_\alpha \in \Gamma^\circ$. Therefore $W \subset \Gamma^\circ$. Suppose there exists a proper sub-root system Ψ' of Ψ'_T containing X . For any closed connected subgroup H of G with T a maximal torus of

H , one has that each root of $\Phi(H, T)$ is contained in X . Hence $\Phi(H, T) \subset X \subset \Psi'$. This contradicts to the condition that the root systems $\Phi(H, T)$ generate Ψ'_T .

By the above two facts and Lemma 3.8 we get $W_{\Psi'_T} = W \subset \Gamma^\circ$. \square

Corollary 3.10. *For any root α in Ψ'_T , there exists a closed connected subgroup H of G with T a maximal torus of H and having root system equal to $\{\pm\alpha\}$. The abstract root system Ψ'_T depends only on the group G and the connected torus T , not on the biinvariant Riemannian metric m .*

Proof. Considering the subset X defined in the above proof for Proposition 3.9, it is obviously Γ° invariant. Since $\Gamma^\circ \supset W_{\Psi'_T}$ as the above proof showed, X is a union of some $W_{\Psi'_T}$ on Ψ'_T . Thus X is a sub-root system of Ψ'_T . On the other hand, the above proof showed that X generates Ψ'_T . Therefore $X = \Psi'_T$. This is just the first statement in the conclusion of the corollary. Moreover, by Proposition 2.6 the cusp product on Ψ'_T is determined. Therefore, the abstract root system Ψ'_T is determined by G and T , independent with the biinvariant Riemannian metric m . \square

Remark 3.11. *Suppose the abstract root system Ψ'_T is given, an interesting question is to determine the sub-root systems Φ of Ψ'_T so that there exists a closed connected subgroup H of G with T a maximal torus of H and having root system equal to Φ . From the above corollary, we know that each rank-1 sub-root system of Ψ'_T is the root system of a closed connected subgroup. For sub-root systems of higher rank, it is unclear to the author which sub-root systems of Ψ'_T are root systems of closed connected subgroups. Moreover, is it possible that any sub-root system of Ψ'_T of rank larger than one is not the root system of a closed connected subgroup?*

Given a reduced root system Φ on T , let

$$\delta_\Phi = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha.$$

Let

$$\delta'_\Phi$$

the unique dominant weight with respect to $(\Psi_T)^+$ in the orbit $W_{\Psi_T} \delta_\Phi = \{\gamma \delta_\Phi : \gamma \in W_{\Psi_T}\}$. Here W_{Ψ_T} is the Weyl group of the root system Ψ_T and $\Phi^+ = \Phi \cap (\Psi_T)^+$.

Definition 3.12. *Define a character on T ,*

$$A_\Phi = \sum_{w \in W_\Phi} \text{sign}(w) [\delta_\Phi - w \delta_\Phi].$$

Moreover, for any finite group W between W_Φ and $\text{Aut}(\Lambda_{\mathbb{Q}, T}, m)$, define another character on T

$$F_{\Phi, W} = \frac{1}{|W|} \sum_{\gamma \in W} \gamma(A_\Phi).$$

Definition 3.13. *Given a finite subgroup W of $\text{Aut}(\Lambda_{\mathbb{Q}, T}, m)$ and an integral weight $\lambda \in \Lambda_T$, let*

$$\chi_{\lambda, W}^* = \frac{1}{|W|} \sum_{\gamma \in W} [\gamma \lambda] \in \mathbb{Q}[\Lambda_{\mathbb{Q}, T}].$$

The character $\chi_{\lambda, W}^*$ depends only on the orbit $W\lambda = \{\gamma \lambda : \gamma \in W\}$. If $W \subset \text{Aut}(\Lambda_T, m)$, then $\chi_{\lambda, W}^* \in \mathbb{Q}[\Lambda_T]$ for any $\lambda \in \Lambda_T$. Moreover, the set $\{\chi_{\lambda, W}^* | \lambda \in \Lambda_T^1\}$ is a basis of $\mathbb{Q}[\Lambda_T]^W$, where Λ_T^1 is a set of representatives of W orbits in Λ_T . In the case

that $\text{rank } \Psi_T = \dim T$ and $W = W_{\Psi_T}$, we can choose $\Lambda_T^1 = \Lambda_T^+$. Here, Λ_T^+ is the set of dominant integral weights in Λ_T with respect to $(\Psi_T)^+$.

The characters $F_{\Phi, W}$ have the following property.

Proposition 3.14. *Each character $F_{\Phi, W}$ is a linear combination of $\{\chi_{\lambda, W}^* \mid \lambda \in \Lambda_{\mathbb{Q}, T}\}$ with integer coefficients and the constant term of $F_{\Phi, W}$ is 1. With $\{\chi_{\lambda, W}^* \mid \lambda \in \Lambda_{\mathbb{Q}, T}\}$ as a basis, the minimal length terms of $F_{\Phi, W} - 1$ are of the form $\{c_\alpha \cdot \chi_{\alpha, W}^* \mid c_\alpha \in \mathbb{Z}, \alpha \in \Phi^\circ\}$ and the unique longest term of $F_{\Phi, W}$ is of the form $c \cdot \chi_{2\delta_\Phi, W}^*$ where $c = \pm 1$.*

Proof. Since W contains W_Φ ,

$$\begin{aligned} F_{\Phi, W} &= \frac{1}{|W|} \sum_{\gamma \in W} \gamma(A_\Phi) \\ &= \frac{1}{|W|} \sum_{\gamma \in W} \sum_{w \in W_\Phi} \text{sign}(w) (\gamma[\delta_\Phi - w\delta_\Phi]) \\ &= \sum_{w \in W_\Phi} \text{sign}(w) \chi_{\delta_\Phi - w\delta_\Phi, W}^*. \end{aligned}$$

One has $|\delta_\Phi - w\delta_\Phi|^2 = \langle 2\delta_\Phi, \delta_\Phi - w\delta_\Phi \rangle$ and

$$\delta_\Phi - w\delta_\Phi = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Phi^+} w\alpha = \sum_{\alpha \in \Phi^+ \cap w^{-1}\Phi^-} \alpha.$$

Then for $w \neq 1$, $\delta_\Phi - w\delta_\Phi$ is of shortest length exactly when $w = s_\alpha$ for α , a short root of minimal length; and it is of longest length exactly when $w\Phi^+ = \Phi^-$, i.e., $w = w_0$ is the unique longest element in W . \square

If $W_{\Psi_T} \subset W$, then $\chi_{2\delta_\Phi, W}^* = \chi_{2\delta'_\Phi, W}^*$. The weight $2\delta'_\Phi$ is defined in [LP] and it is observed there that $\chi_{2\delta'_\Phi, W}^*$ is the unique leading term of $F_{\Phi, W}$ if $W_{\Psi_T} \subset W$.

The following two propositions connect Sato-Tate measures st_H and the characters $\{F_{\Phi, \Gamma^\circ}\}$. The importance of them is: we are able to tackle dimension data by studying the algebraic objects F_{Φ, Γ° . Proposition 3.16 reduces the dimension datum problem and the linear dependence problem to comparing the characters $\{F_{\Phi, \Gamma^\circ} \mid \Phi \subset \Psi_T\}$ and getting linear relations among them. After this, combinatorial classification of reduced sub-root systems and algebraic method treating these characters come into force.

The proof of the following proposition can be found in [LP], Section 1 or [AYY], Section 4. For completeness, we give a sketch of the proof.

Proposition 3.15. *For a closed subgroup H of G , the support of the Sato-Tate measure st_H is the set of conjugacy classes contained in the set $\{gxg^{-1} \mid g \in G, x \in H\}$. If H is connected, then for a maximal torus T of H contained in a maximal torus S of G , one has*

$$\left(\bigcup_{w=nS \in W_G} nTn^{-1} \right) / W_G \subset S / W_G \subset G^\natural,$$

S/W_G being a connected component of G^\natural and $(\bigcup_{w=nS \in W_G} nTn^{-1})/W_G$ being equal to the support of st_H . Again if H is connected, then for the natural map $\pi : T \rightarrow \text{supp}(\text{st}_H)$, one has

$$\pi^*(\text{st}_H) = F_{\Phi, \Gamma^\circ}(t),$$

where $\Phi = \Phi(H)$.

Proof. The first and the second statements are clear.

Given H being connected, for a conjugation invariant continuous function f on G , one has

$$\int_H f d\mu_H = \int_H f d\mu_H.$$

By the Weyl integration formula,

$$\int_H f d\mu_H = \frac{1}{|W_H|} \int_T f(t) F_\Phi(t) dt,$$

where $F_\Phi(t)$ is the Weyl product. Writing this expression Γ° invariant, we get

$$\begin{aligned} & \frac{1}{|W_H|} \int_T f(t) F_\Phi(t) dt \\ &= \frac{1}{|W_H|} \frac{1}{|Stab_{W_G}(T)|} \sum_{\gamma = nS \in W_G, nTn^{-1} = T} \int_T f(n^{-1}tn) F_\Phi(n^{-1}tn) dt \\ &= \frac{1}{|W_H|} \frac{1}{|Stab_{W_G}(T)|} \sum_{\gamma = nS \in W_G, nTn^{-1} = T} \int_T f(t) F_\Phi(n^{-1}tn) dt \\ &= \frac{1}{|W_H|} \frac{1}{|\Gamma^\circ|} \int_T f(t) \left(\sum_{\gamma \in \Gamma^\circ} F_\Phi(\gamma^{-1}(t)) \right) dt. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & F_\Phi(t) \\ &= \prod_{\alpha \in \Phi} (1 - [\alpha]) \\ &= \prod_{\alpha \in \Phi^+} \left(\left[\frac{-\alpha}{2} \right] - \left[\frac{\alpha}{2} \right] \right) \left(\left[\frac{\alpha}{2} \right] - \left[\frac{-\alpha}{2} \right] \right) \\ &= \left(\sum_{w \in W_\Phi} \text{sgn}(w) [-w\delta] \right) \left(\sum_{w \in W_\Phi} \text{sgn}(w) [w\delta] \right) \\ &= \sum_{w, \tau \in W} \text{sgn}(w) \text{sgn}(\tau) [-w\delta + \tau\delta] \\ &= \sum_{w, \tau \in W_\Phi} \text{sgn}(w) [-\tau w\delta + \tau\delta] \quad (\text{use } w \rightarrow \tau w) \\ &= \sum_{\tau \in W_\Phi} \tau \left(\sum_{w \in W_\Phi} \text{sgn}(w) [\delta - w\delta] \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{|W_H|} \frac{1}{|\Gamma^\circ|} \sum_{\gamma \in \Gamma^\circ} F_\Phi(\gamma^{-1}(t)) \\ &= \frac{1}{|W_H|} \frac{1}{|\Gamma^\circ|} \sum_{\gamma \in \Gamma^\circ} \gamma(F_\Phi(t)) \\ &= \frac{1}{|W_H|} \frac{1}{|\Gamma^\circ|} \sum_{\gamma \in \Gamma^\circ} \gamma \left(\sum_{\tau \in W_\Phi} \tau \left(\sum_{w \in W_\Phi} \text{sgn}(w) [\delta - w\delta] \right) \right) \\ &= \frac{1}{|\Gamma^\circ|} \sum_{\gamma \in \Gamma^\circ} \gamma \left(\sum_{w \in W_\Phi} \text{sgn}(w) [\delta - w\delta] \right) \\ &= F_{\Phi, \Gamma^\circ}(t). \end{aligned}$$

Hence the last statement follows. \square

The proof of the following proposition is a bit complicated. The basic idea is as follows. Given a maximal torus S of a connected compact Lie group G and $W_G = N_G(S)/C_G(S)$, then the set G^\natural of conjugacy classes in G can be identified with S/W_G . For any connected closed subgroup H with a maximal torus T contained in S , the support of the Sato-Tate measure st_H of H is $p(T)$ where $p : S \rightarrow S/W_G$ be the natural projection. This indicates that: not only $\text{supp}(\text{st}_H)$ has dimension equal to $\dim T$, but also the integral of st_H against a continuous function on S/W_G with a given bound and support near a given set of dimension less than $\dim T$ can be made arbitrarily small if the distance of the support of the function to that given set is sufficiently small. With this fact, by choosing test functions on S/W_G appropriately we are able to distinguish different Sato-Tate measures by their supports.

Proposition 3.16. *Given a list $\{H_1, H_2, \dots, H_s\}$ of connected closed subgroups of a compact Lie group G and non-zero real numbers c_1, \dots, c_s ,*

$$\sum_{1 \leq i \leq s} c_i \mathcal{D}_{H_i} = 0$$

if and only if for any closed connected torus T in G ,

$$\sum_{1 \leq j \leq t} c_{i_j} F_{\Phi_{i_j}, \Gamma^\circ} = 0,$$

where $\{H_{i_j} | i_1 < i_2 < \dots < i_t\}$ are all subgroups among $\{H_i | 1 \leq i \leq s\}$ with maximal tori conjugate to T in G , Φ_{i_j} is the root system of H_{i_j} regarded as a root system on T and $\Gamma^\circ = N_G(T)/C_G(T)$.

Proof. By the relation between dimension data and Sato-Tate measures, the equality

$$\sum_{1 \leq i \leq s} c_i \mathcal{D}_{H_i} = 0$$

is equivalent to $\sum_{1 \leq i \leq s} c_i \text{st}_{H_i} = 0$.

Choosing a maximal torus T_i of G_i , for any $1 \leq i \leq s$, we may and do assume that $\{T_i : 1 \leq i \leq s\}$ are all contained in a maximal torus S of G . The set of conjugacy classes of G contained in G_0 , $G_0/\text{Ad}(G) \subset G^\natural$ can be identified with S/W_G . Let $p : S \rightarrow S/W_G$ be the natural projection. Under the above identification, the support of st_{H_i} is $p(T_i)$. Among $\{T_i : 1 \leq i \leq s\}$, we may assume that $T = T_1 = T_2 = \dots = T_t$, and any other T_i ($i \geq t+1$) is not conjugate to T and has dimension $\geq \dim T$.

The Riemannian metric m on G induces a Riemannian metric on S . It gives S the structure of a metric space. Since W_G acts on S by isometries, S/W_G inherits a metric structure. For any $i \geq t+1$, since T_i is not conjugate to T and has dimension $\geq \dim T$, $gT_i g^{-1} \not\subset T$ for any $g \in G$. Moreover each T_i is a closed torus, thus $p(T) \cap p(T_i) \subset p(T_i)$ is a union of the images under p of finitely many closed tori of S of strictly lower dimension than $\dim T_i$.

Given any $\epsilon > 0$, let \mathcal{U}_ϵ be the subset of S/W_G consisting of points with distance to $p(T)$ within ϵ . For any continuous function f on S/W_G with absolute value bounded by a positive number $K/2$, there exists a continuous function f_ϵ on S/W_G with support contained in \mathcal{U}_ϵ and absolute value bounded by K , and having restriction to $p(T)$ equal to $f|_T$. Since $\sum_{1 \leq i \leq s} c_i \text{st}_{H_i} = 0$,

$$(2) \quad \sum_{1 \leq i \leq s} c_i \text{st}_{H_i}(f_\epsilon) = 0.$$

For any $i \geq t + 1$, since $p(T) \cap p(T_i) \subset p(T_i)$ is a union of the images under p of finitely many closed tori of S of strictly lower dimension than $\dim T_i$ and st_{H_i} has support equal to $p(T_i)$, we get

$$\lim_{\epsilon \rightarrow 0} \text{st}_{H_i}(f_\epsilon) = 0.$$

Taking the limit as ϵ approaches 0 in Equation (2), we get

$$\sum_{1 \leq i \leq t} c_i \text{st}_{H_i}(f) = 0.$$

Hence

$$\sum_{1 \leq i \leq t} c_i \text{st}_{H_i} = 0.$$

Therefore an inductive argument on s finishes the proof. \square

Remark 3.17. *In Proposition 3.16, if G is a connected compact simple Lie group with a root system Ψ , then $\Gamma^\circ = W_\Psi$. Moreover, in this case the conjugacy class of a full rank subgroup connected closed subgroup is determined by its root system, which can be regarded as a sub-root system of Ψ . In this way, finding linear relations among dimension data of connected full rank subgroups is equivalent to finding linear relations among the characters $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$.*

4. COMPARISON OF DIFFERENT CONJUGACY CONDITIONS

In this section we discuss relations among several finite groups Γ° , Γ , $W_{\Psi'_T}$, W_{Ψ_T} and $\text{Aut}(\Psi_T)$ defined in previous sections. Moreover, we discuss the connection between various relations of two subgroups and conjugacy relations of their root-systems with respect to these finite groups.

Given a compact Lie group G with a biinvariant Riemannian metric m and a closed connected torus T , denote by $\Lambda_T = \text{Hom}(T, \text{U}(1))$. We have defined two finite groups Γ° , Γ by

$$\Gamma^\circ = N_G(T)/C_G(T)$$

and

$$\Gamma = \text{Aut}(\Lambda_T, m|_{\Lambda_T}).$$

Moreover, we get a root system Ψ_T on T . Thus we have the finite groups W_{Ψ_T} and $\text{Aut}(\Psi_T)$. By Definition 3.3 the action of W_{Ψ_T} stabilizes Λ_T and preserves $m|_{\Lambda_T}$. Thus W_{Ψ_T} can be regarded as a subgroup of $\Gamma = \text{Aut}(\Lambda_T, m|_{\Lambda_T})$. By the definition of Ψ_T it is stable under the action of Γ . Thus there is a group homomorphism

$$\pi : \Gamma \longrightarrow \text{Aut}(\Psi_T).$$

In the case that $\text{rank } \Psi_T = \dim T$, π is an injective map. In general it is not injective, however its restriction to W_{Ψ_T} is injective.

Proposition 4.1. *In general the inclusions $\Gamma^\circ \subset \Gamma$, $W_{\Psi_T} \subset \Gamma$ and $\pi(\Gamma) \subset \text{Aut}(\Psi_T)$ are proper inclusions and neither Γ° nor W_{Ψ_T} contain the other one.*

Proof. Given $n \geq 5$, denote by $G = \text{Aut}(\mathfrak{su}(\mathfrak{n}))$ and T a maximal torus of G . Then we have $\Psi_T = A_{n-1}$, $\text{rank } \Psi_T = \dim T$, $\Gamma^\circ = \Gamma = \text{Aut}(\Psi_T) = S_n \times \{-1\}$ and $W_{\Psi_T} = S_n$. In this case $\Gamma \neq W_{\Psi_T}$ and $\Gamma^\circ \not\subset W_{\Psi_T}$.

In Example 10.4, by choosing W appropriately we have $\Gamma \neq \Gamma^\circ$ and $W_{\Psi_T} \not\subset \Gamma^\circ$.

Let $G = (\text{SU}(2)^3)/\langle(-I, -I, I)\rangle$ and T a maximal torus of G . In this case $\Psi_T = 3A_1$ and $\text{rank } \Psi_T = \dim T$. Thus π is injective. Moreover, we have

$$\Gamma = \text{Aut}(T, m|_T) = \{\pm 1\}^3 \rtimes \langle \sigma_{12} \rangle$$

and

$$\mathrm{Aut}(\Psi_T) = \{\pm 1\}^3 \rtimes S_3,$$

where σ_{12} is the transposition on the first and the second positions. Therefore $\pi(\Gamma) \neq \mathrm{Aut}(\Psi_T)$. \square

Definition 4.2. *Given two compact Lie groups H and G , two homomorphisms $\phi_1, \phi_2 : H \rightarrow G$ are said element-conjugate if $\phi_1(x) \sim \phi_2(x)$ for any $x \in H$. Similarly, we call two closed subgroups H_1, H_2 of G element-conjugate if there exists an isomorphism $\phi : H_1 \rightarrow H_2$ such that $x \sim \phi(x)$ for any $x \in H$.*

Element-conjugate homomorphisms are defined and studied in [Lar]. It is proved in [Lar] and [Lar2] that for G equal to $\mathrm{SU}(n)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(n)$ or G_2 , element-conjugate homomorphisms to G are actually conjugate. In the converse direction, for G equal to $\mathrm{SO}(2n)$ ($n \geq 4$) or a connected simple group of type E_6 , E_7 , E_8 or F_4 , there exist element-conjugate homomorphisms to G which are not conjugate. In [W], S. Wang considered homomorphisms from connected groups and used the name locally conjugate instead of element-conjugate. He gave some examples of element-conjugate homomorphisms from connected groups to $\mathrm{SO}(2n)$ which are not conjugate. In the 1950s Dynkin classified semisimple subalgebras of complex simple Lie algebras up to *linear conjugacy* (cf. [Dy]). Moreover, Minchenko distinguished the conjugacy classes among linear conjugacy classes of semisimple subalgebras (cf. [M]). By [M] Theorem 1 and Proposition 4.3 below, two connected closed subgroups are element-conjugate if and only if their subalgebras are linear conjugate. From this, connected closed subgroups of connected compact simple Lie group which are locally conjugate but not globally conjugate can be classified as well.

Proposition 4.3. *Given two compact Lie groups H, G with H connected, and a maximal torus T of H , two homomorphisms $\phi_1, \phi_2 : H \rightarrow G$ are locally conjugate if and only if there exists $g \in G$ such that $\phi_2|_T = (\mathrm{Ad}(g) \circ \phi_1)|_T$ and the root systems $\Phi(H_1, \phi_1(T)) = \Phi(H_2, \phi_1(T))$, where we denote by $H_1 = (\mathrm{Ad}(g) \circ \phi_1)(H)$ and $H_2 = \phi_2(H)$.*

Proof. The “if” part is clear. We prove the “only if” part. Choosing a maximal torus S of G , write $W(G, S) = N_G(S)/C_G(S)$. Since ϕ_1, ϕ_2 are element-conjugate, we have $\ker \phi_1 = \ker \phi_2$. Without loss of generality we may assume that ϕ_1, ϕ_2 are injective and $\phi_1(T), \phi_2(T) \subset S$.

For any $x \in T$, since $\phi_1(x) \sim \phi_2(x)$, we have $\phi_2 = n(\phi_1(x))n^{-1}$ for some $w = nS \in W(G, S)$. As $W(G, S)$ is finite, there exists $w \in W(G, S)$ such that $\phi_2 = n(\phi_1(x))n^{-1}$ holds for a set of $x \in T$ generating T . Hence $\phi_2 = n(\phi_1(x))n^{-1}$ for any $x \in T$. This means that $(\mathrm{Ad}(n) \circ \phi_1)|_T = \phi_2|_T$. For simplicity, we assume that $\phi_1|_T = \phi_2|_T$. We still denote by T the image in S of T under ϕ_1 and ϕ_2 . In this way $\phi_1|_T$ and $\phi_2|_T$ are both the identity map. Write

$$\mathfrak{h} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{h}_\alpha \right)$$

for the root space decomposition of $\mathfrak{h} = (\mathrm{Lie} H) \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the T action and

$$\mathfrak{g} = \mathfrak{g}^T \oplus \left(\bigoplus_{\lambda \in \Lambda_T} \mathfrak{g}_\lambda \right)$$

the generalized root space decomposition of $\mathfrak{g} = (\mathrm{Lie} G) \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the T action, where $\Phi = \Phi(H, T)$ is root system of H and $\Lambda_T = \mathrm{Hom}(T, \mathrm{U}(1))$ is the integral weight lattice. As $\phi_1|_T$ and $\phi_2|_T$ are both the identity map,

$$\phi_1(\mathfrak{h}_\alpha), \phi_2(\mathfrak{h}_\alpha) \subset \mathfrak{g}_\alpha$$

for any $\alpha \in \Phi$. That just means $\Phi(H_1, \phi_1(T)) = \Phi(H_2, \phi_1(T))$. \square

Given a compact Lie group G with a biinvariant Riemannian metric m and a connected closed torus T , denote by H_1, H_2 two connected closed subgroups of G , we could consider different conjugacy relations between their root systems. Precisely, write $\Phi_i = \Phi(H_i, T)$ for the root system of H_i , $i = 1, 2$. We may consider if $\Phi_1, \Phi_2 \subset \Psi_T \subset \Lambda_T$ are conjugate under $\Gamma^\circ, \Gamma, W_\Psi$ or $\text{Aut}(\Psi_T)$. Given two root systems $\Phi_1, \Phi_2 \subset \Psi_T$, since $\Gamma^\circ \subset \Gamma$ and $\pi(\Gamma) \subset \text{Aut}(\Psi_T)$, we have

$$\Phi_1 \sim_{\Gamma^\circ} \Phi_2 \implies \Phi_1 \sim_\Gamma \Phi_2 \implies \Phi_1 \sim_{\text{Aut}(\Psi_T)} \Phi_2.$$

We have the following connections between the conjugacy relations between the root systems and the relations between the subgroups.

Proposition 4.4. *Given a compact Lie group G with a biinvariant Riemannian metric m and a connected closed torus T , for two closed subgroups H_1, H_2 with T a common maximal torus of them, denote by $\Phi_i = \Phi(H_i, T)$, $i = 1, 2$. We have:*

- (1) *if $\Phi_1 \sim_{\text{Aut}(\Psi_T)} \Phi_2$, then the Lie algebras of H_1 and H_2 are isomorphic.*
- (2) *If $\Phi_1 \sim_\Gamma \Phi_2$, then the Lie groups H_1 and H_2 are isomorphic.*
- (3) *$\Phi_1 \sim_{\Gamma^\circ} \Phi_2$ if and only if H_1 and H_2 are locally conjugate.*

Proof. Write

$$\mathfrak{g} = \mathfrak{g}^T \oplus \left(\bigoplus_{\lambda \in \Lambda_T} \mathfrak{g}_\lambda \right)$$

for the generalized root space decomposition of $\mathfrak{g} = (\text{Lie } G) \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the T action and

$$\mathfrak{h}_i = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi_i} \mathfrak{h}_{i,\alpha} \right)$$

the root space decomposition of $\mathfrak{h}_i = (\text{Lie } H_i) \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the T action, $i = 1, 2$. We have $\Phi_1, \Phi_2 \subset \Psi_T \subset \Lambda_T$.

Suppose $\Phi_1 \sim_{\text{Aut}(\Psi_T)} \Phi_2$. This means that there exists $\gamma \in \text{Aut}(\Psi_T)$ such that $\Phi_2 = \gamma\Phi_1$. By the classification of complex reductive Lie algebras (cf. [Kn]), γ can be lifted to an isomorphism $f : \mathfrak{h}_1 \longrightarrow \mathfrak{h}_2$ such that $f(\mathfrak{h}_{1,\alpha}) = \mathfrak{h}_{2,\gamma\alpha}$ for any $\alpha \in \Phi_1$. This proves (1).

Suppose $\Phi_1 \sim_\Gamma \Phi_2$. Recall that $\Gamma = \text{Hom}(T, m|_T)$. For the automorphism f in the above proof, furthermore we may assume that $f|_{\text{Lie } T}$ is induced by an automorphism of T . Thus f can be lifted to an isomorphism $f : H_1 \longrightarrow H_2$. This proves (2).

Recall that $\Gamma^\circ = N_G(T)/C_G(T)$. If $\Phi_1 \sim_{\Gamma^\circ} \Phi_2$, then $\Phi_1 = \Phi'_2$ for some $H'_2 = gH_2g^{-1}$, $g \in N_G(T)$ and $\Phi'_2 = \Phi(H'_2, T)$. Therefore (3) follows from Proposition 4.3. \square

Remark 4.5. *Given a compact Lie group G and a connected closed torus T , as indicated in the proof of Proposition 4.3, a trivial observation is: the existence of locally-conjugate but not globally conjugate connected closed subgroups with T a maximal torus of them is due to the fact the generalized root spaces $\{\mathfrak{g}_\lambda : \lambda \in \Lambda_T\}$ may have dimension larger than one and the different choices of eigenvectors really matter for the conjugacy classes of the subgroups. Hence there is no local-global issue if each \mathfrak{g}_α having dimension one. In particular, two full rank connected subgroups are conjugate if and only if they are locally conjugate.*

5. FORMULATION OF THE PROBLEMS IN TERMS OF ROOT SYSTEMS

We formulate the following two questions, analogous to the dimension datum problem and the linear dependence problem respectively.

Question 5.1. *Given a root system Ψ , for which pairs (Φ_1, Φ_2) of reduced sub-root systems of Ψ , we have $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$?*

Question 5.2. *Given a root system Ψ and a finite group W such that $W_\Psi \subset W \subset \text{Aut}(\Psi)$, for which reduced sub-root systems $\Phi_1, \Phi_2, \dots, \Phi_s$ of Ψ , the characters*

$$F_{\Phi_1, W}, F_{\Phi_2, W}, \dots, F_{\Phi_s, W}$$

are linearly dependent?

Remark 5.1. *We could take $(\Psi, W) = (\Psi_T, \pi(\Gamma))$ or $(\Psi, W) = (\Psi'_T, \pi(\Gamma^\circ))$ in Question 5.2. The property $W_{\Psi_T} \subset \pi(\Gamma) \subset \text{Aut}(\Psi_T)$ follows from the definitions of Γ and of Ψ_T . The property $W_{\Psi'_T} \subset \pi(\Gamma^\circ) \subset \text{Aut}(\Psi'_T)$ is proved in Proposition 3.9. Taking the pair (Ψ'_T, Γ°) , the solution to Question 5.2 gives a solution to Question 1.2.*

6. CLASSIFICATION OF SUB-ROOT SYSTEMS

In this section, given an irreducible root system Ψ_0 , we discuss the classification of sub-root systems of Ψ_0 up to W_{Ψ_0} conjugation. In the case that Ψ_0 is a classical irreducible root system, classification of sub-root systems of it seems to have been known to experts long time ago. A detailed exposition of this classification is given in [Os]. In the case that Ψ_0 is an exceptional irreducible root system, classification of sub-root systems of it was first achieved by Oshima in [Os] except that the classification of sub-root systems of F_4 or G_2 was known before. Our discussion follows [LP] and [Os].

Before discussing the classification, we fix notations. Let $V = \mathbb{R}^n$ be an n -dimensional Euclidean vector space with an orthonormal basis $\{e_1, e_2, \dots, e_n\}$. Denote by the root systems

$$\begin{aligned} A_{n-1} &= \{ \pm(e_i - e_j) \mid 1 \leq i < j \leq n \}, \\ B_n &= \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm e_i \mid 1 \leq i \leq n \}, \\ C_n &= \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm 2e_i \mid 1 \leq i \leq n \}, \\ BC_n &= \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \} \cup \{ \pm e_i, \pm 2e_i \mid 1 \leq i \leq n \}, \\ D_n &= \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq n \}. \end{aligned}$$

Thus

$$A_{n-1} \subset D_n \subset B_n, C_n \subset BC_n.$$

Note that $A_0 = D_1 = \emptyset$ and we regard A_1, B_1, C_1 different from each other though they are isomorphic as abstract root systems. Each of the following pairs (B_2, C_2) , $(D_2, 2A_1)$, (A_3, D_3) are also regarded different.

Type A. Given $\Psi_0 = A_{n-1}$ ($n \geq 2$), let $\Phi \subset \Psi_0$ be a sub-root system. Define a relation on $\{1, 2, \dots, n\}$ by

$$i \sim j \Leftrightarrow e_i - e_j \in \Phi.$$

Thus \sim is an equivalence relation. It divides $\{1, 2, \dots, n\}$ into l subsets of cardinalities n_1, n_2, \dots, n_l where $n_1 \geq n_2 \geq \dots \geq n_l$ and $n_1 + n_2 + \dots + n_l = n$. In this way we can show

$$\Phi \sim \bigcup_{1 \leq k \leq l} \{ \pm(e_i - e_j) \mid n_1 + n_2 + \dots + n_{k-1} + 1 \leq i < j \leq n_1 + n_2 + \dots + n_k \}.$$

We denote by

$$\Phi_{n_1, n_2, \dots, n_l} = \bigcup_{1 \leq k \leq l} \{ \pm(e_i - e_j) \mid n_1 + n_2 + \dots + n_{k-1} + 1 \leq i < j \leq n_1 + n_2 + \dots + n_k \}.$$

It is isomorphic to $\bigcup_{1 \leq k \leq l} A_{n_k-1}$.

Type B. Given $\Psi_0 = B_n$ ($n \geq 1$), let $\Phi \subset \Psi_0$ be a sub-root system. Define a relation on $\{1, 2, \dots, r\}$ by

$$i \sim j \Leftrightarrow e_i - e_j \in \Phi \text{ or } e_i + e_j \in \Phi.$$

Thus \sim is an equivalent relation. It divides $\{1, 2, \dots, n\}$ into l subsets of cardinalities n_1, n_2, \dots, n_l where $n_1 + n_2 + \dots + n_l = n$. Write $n'_k = n_1 + n_2 + \dots + n_k$ for $1 \leq k \leq l$. Without loss of generality we may assume that $\{i \mid n'_{k-1} + 1 \leq i \leq n'_k\}$ is an equivalence set for any $1 \leq k \leq l$ and

$$\{\pm(e_i - e_j) \mid n'_{k-1} + 1 \leq i < j \leq n'_k\} \subset \Phi.$$

If $e_i \in \Phi$ for some $n'_{k-1} + 1 \leq i \leq n_k$, we have

$$B_{n_k} \cong \{\pm e_i \pm e_j \mid n'_{k-1} + 1 \leq i < j \leq n'_k\} \cup \{\pm e_i \mid n'_{k-1} + 1 \leq i \leq n'_k\} \subset \Phi.$$

If $e_i \notin \Phi$ for any $n'_{k-1} + 1 \leq i \leq n_k$ and $e_i + e_j \in \Phi$ for some $n'_{k-1} + 1 \leq i < j \leq n'_k$, we have

$$D_{n_k} \cong \{\pm e_i \pm e_j \mid n'_{k-1} + 1 \leq i < j \leq n'_k\} \subset \Phi.$$

In summary we have

$$\Phi \cong \left(\bigcup_{1 \leq i \leq u} B_{r_i} \right) \cup \left(\bigcup_{1 \leq j \leq v} D_{s_j} \right) \cup \left(\bigcup_{1 \leq k \leq w} A_{t_k-1} \right)$$

where $\sum_{1 \leq i \leq u} r_i + \sum_{1 \leq j \leq v} s_j + \sum_{1 \leq k \leq w} t_k = n$, $r_i \geq 1$, $s_j \geq 2$ and $t_k \geq 1$. Moreover, the conjugacy class of Φ is uniquely determined by these indices $\{r_i, s_j, t_k\}$. We denote by $\Phi_{\{r_i\}, \{s_j\}, \{t_k\}}$ a sub-root system with the indices $\{r_i, s_j, t_k\}$.

Type C. Given $\Psi_0 = C_n$, let $\Phi \subset \Psi_0$ be a sub-root system. Similar as in the B_n case, we can show

$$\Phi \cong \left(\bigcup_{1 \leq i \leq u} C_{r_i} \right) \cup \left(\bigcup_{1 \leq j \leq v} D_{s_j} \right) \cup \left(\bigcup_{1 \leq k \leq w} A_{t_k-1} \right)$$

where $\sum_{1 \leq i \leq u} r_i + \sum_{1 \leq j \leq v} s_j + \sum_{1 \leq k \leq w} t_k = n$, $r_i \geq 1$, $s_j \geq 2$ and $t_k \geq 1$. Moreover, the conjugacy class of Φ is uniquely determined by these indices $\{r_i, s_j, t_k\}$. We denote by $\Phi_{\{r_i\}, \{s_j\}, \{t_k\}}$ a sub-root system with the indices $\{r_i, s_j, t_k\}$.

Type BC. Given $\Psi_0 = BC_n$, let $\Phi \subset \Psi_0$ be a sub-root system. Similar as in the B_n case, we can show

$$\Phi \cong \left(\bigcup_{1 \leq i \leq u} BC_{r_i} \right) \cup \left(\bigcup_{1 \leq j \leq v} B_{s_j} \right) \cup \left(\bigcup_{1 \leq k \leq w} C_{t_k} \right) \cup \left(\bigcup_{1 \leq e \leq x} D_{p_e} \right) \cup \left(\bigcup_{1 \leq f \leq y} A_{q_f-1} \right)$$

where $\sum_{1 \leq i \leq u} r_i + \sum_{1 \leq j \leq v} s_j + \sum_{1 \leq k \leq w} t_k + \sum_{1 \leq e \leq x} p_e + \sum_{1 \leq f \leq y} q_f = n$, $r_i \geq 1$, $s_j \geq 1$, $t_k \geq 1$, $p_e \geq 2$ and $q_f \geq 1$. Moreover, the conjugacy class of Φ is uniquely determined by the indices $\{r_i, s_j, t_k, p_e, q_f\}$. We denote by $\Phi_{\{r_i\}, \{s_j\}, \{t_k\}, \{p_e\}, \{q_f\}}$ a sub-root system with the indices $\{r_i, s_j, t_k, p_e, q_f\}$.

Type D. Given $\Psi_0 = D_n$ ($n \geq 5$), in this case $\text{Aut}(\Psi_0) = W_n = \{\pm 1\}^n \rtimes S_n$ and W_{Ψ_0} is a subgroup of index 2 in W_n . Let $\Phi \subset \Psi_0$ be a sub-root system. Similar as in B_n case, we have show

$$\Phi \cong \left(\bigcup_{1 \leq j \leq v} D_{s_j} \right) \cup \left(\bigcup_{1 \leq k \leq w} A_{t_k-1} \right)$$

where $\sum_{1 \leq j \leq v} s_j + \sum_{1 \leq k \leq w} t_k = n$, $s_j \geq 2$ and $t_k \geq 1$. Moreover, the $\text{Aut}(\Psi_0)$ conjugacy class of Φ is uniquely determined by the indices $\{s_j, t_k\}$. Given a set of indices $\{s_j, t_k : 1 \leq j \leq v, 1 \leq k \leq w\}$, there are at most two conjugacy classes of sub-root systems up to W_{Ψ_0} conjugation. In the case that there is a unique W_{Ψ_0} conjugacy class of sub-root

systems with this set of indices, we denote by $\Phi_{\{s_j\}, \{t_k\}}$ one of them. Otherwise, we can show that each $s_j = 0$ and each t_k is even. Write

$$\Phi_{\{t_k\}} = \bigcup_{1 \leq k \leq l} \{ \pm (e_i - e_j) \mid t_1 + t_2 + \cdots + t_{k-1} + 1 \leq i < j \leq t_1 + t_2 + \cdots + t_k \}$$

and $\Phi'_{\{t_k\}} = s_{e_1} \Phi_{\{t_k\}}$. They represent these two conjugacy classes.

Type D₄. Given $\Psi_0 = D_4 = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq 4\}$, we denote by

$$\begin{aligned} A_1 &= \langle e_1 - e_2 \rangle, \\ A_2 &= \langle e_1 - e_2, e_2 - e_3 \rangle, \\ A_3 &= \langle e_1 - e_2, e_2 - e_3, e_3 - e_4 \rangle, \\ A'_3 &= \langle e_1 - e_2, e_2 - e_3, e_3 + e_4 \rangle, \\ 2A_1 &= \langle e_1 - e_2, e_3 - e_4 \rangle, \\ 2A'_1 &= \langle e_1 - e_2, e_3 + e_4 \rangle, \\ D_2 &= \langle e_1 - e_2, e_1 + e_2 \rangle, \\ 3A_1 &= \langle e_1 - e_2, e_1 + e_2, e_3 - e_4 \rangle, \\ 2D_2 &= \langle e_1 - e_1, e_1 + e_2, e_3 - e_4, e_3 + e_4 \rangle, \\ D_3 &= \langle e_1 - e_2, e_2 - e_3, e_2 + e_3 \rangle, \\ D_4 &= \langle e_1 - e_1, e_2 - e_3, e_3 - e_4, e_3 + e_4 \rangle. \end{aligned}$$

We can show that any sub-root system of D_4 is conjugate to one of them up to W_{Ψ_0} conjugation. On the other hand, we have

$$\begin{aligned} A_3 &\sim_{\text{Aut}(D_4)} A'_3 \sim_{\text{Aut}(D_4)} D_3, \\ 2A_1 &\sim_{\text{Aut}(D_4)} 2A'_1 \sim_{\text{Aut}(D_4)} D_2. \end{aligned}$$

These are all $\text{Aut}(D_4)$ conjugacy relations among them.

Type E₆, E₇ and E₈. Given $\Psi_0 = E_6, E_7$ or E_8 , the classification of sub-root systems of Ψ_0 is hard to describe. The readers can refer [Os] for the details.

Type F₄. Given $\Psi_0 = F_4$, we denote by $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ a simple system. The long roots and short roots in Ψ_0 consist in the sub-root systems D_4^L and D_4^S , respectively. There is a unique conjugacy class of sub-root systems of F_4 isomorphic to B_4 (or C_4). We denote by

$$B_4 = \langle \alpha_2, \alpha_1, \alpha_2 + 2\alpha_3, \alpha_4 \rangle$$

and

$$C_4 = \langle \alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_1 \rangle,$$

which are representatives of them. The sub-root systems D_4^L and D_4^S are stable under W_{F_4} . Hence there exist homomorphisms

$$W_{F_4} \longrightarrow \text{Aut}(D_4^L)$$

and

$$W_{F_4} \longrightarrow \text{Aut}(D_4^S).$$

Lemma 6.1. *Each of these two homomorphisms is an isomorphism.*

Proof. We show $W_{F_4} \longrightarrow \text{Aut}(D_4^L)$ is an isomorphism. The proof for $W_{F_4} \longrightarrow \text{Aut}(D_4^S)$ is an isomorphism is similar. Suppose $w \in W_{F_4}$ is an element with trivial restriction on D_4^L . Choose an element α in D_4^S . Since α and D_4^L generate a sub-root system isomorphic to B_4 , we have $\alpha = \frac{\beta_1 + \beta_2}{2}$ for some $\beta_1, \beta_2 \in D_4^L$. Thus

$$w(\alpha) = \frac{w\beta_1 + w\beta_2}{2} = \frac{\beta_1 + \beta_2}{2} = \alpha.$$

Hence $w = 1$. Therefore $W_{F_4} \longrightarrow \text{Aut}(D_4^L)$ is injective. On the other hand, by calculating stabilizers we get

$$|W_{F_4}| = 24|W_{B_3}| = 24 \times 3! \times 2^3 = 3 \times 4! \times 2^4 = |\text{Aut}(D_4)|.$$

Hence $W_{F_4} \longrightarrow \text{Aut}(D_4^L)$ is an isomorphism. \square

By the above lemma and the classification in D_4 case, we get

$$\begin{aligned} D_2^L &\sim 2A_1^L \sim 2A_1'^L, \\ D_3^L &\sim A_3^L \sim A_3'^L, \\ D_2^S &\sim 2A_1^S \sim 2A_1'^S, \\ D_3^S &\sim A_3^S \sim A_3'^S. \end{aligned}$$

The following sub-root systems of F_4 are contained in B_4 :

$$\begin{aligned} &D_4^L, A_3^L + A_1^S, 4A_1^L, B_2 + 2A_1^L, 2A_1^L + 2A_1^S, \\ &B_4, B_3 + A_1^S, 2B_2, B_2 + 2A_1^S, 4A_1^S, A_3^L, \\ &2A_1^L + A_1^S, B_3, B_2 + A_1^S, 3A_1^S, 3A_1^L, \\ &B_2 + A_1^L, A_1^L + 2A_1^S, A_2^L + A_1^S, 2A_1^L, \\ &B_2, 2A_1^S, A_1^L + A_1^S, A_2^L, A_1^S, A_1^L. \end{aligned}$$

By duality, we get sub-root systems of F_4 contained in some C_4 . In particular, the following are those contained in C_4 and not contained in any B_4 :

$$\begin{aligned} &D_4^S, A_1^L + A_3^S, C_4, C_3 + A_1^L, \\ &A_3^S, C_3, A_1^L + A_2^S, A_2^S. \end{aligned}$$

Denote by $\Phi \subset F_4$ a sub-root system not contained in any B_4 or C_4 . If Φ is simple, then $\Phi = F_4$. If Φ is not simple, it can not contain a factor A_1^L, A_1^S or B_2 since otherwise Φ is contained in a sub-root system isomorphic to B_4 or C_4 . Hence $\Phi = A_2^L + A_2^S$. Moreover, there exists a unique conjugacy class of sub-root systems in each of the above types.

Type G_2 . Given $\Psi_0 = G_2$, denote by $\{\alpha, \beta\}$ a simple system of Ψ_0 . We denote by

$$A_1^L = \langle \alpha \rangle, A_1^S = \langle \beta \rangle, A_2^L = \langle \alpha, \alpha + 3\beta \rangle, A_2^S = \langle \alpha + \beta, \beta \rangle$$

and

$$A_1^L + A_1^S = \langle \alpha, \alpha + 2\beta \rangle, G_2 = \langle \alpha, \beta \rangle.$$

Therefore each sub-root system of G_2 is conjugate to one of them.

7. FORMULAS OF THE LEADING TERMS

Given an irreducible root system Ψ_0 with a positive system Ψ_0^+ , we normalize the inner product on Ψ_0 by letting the short roots having length 1.

Definition 7.1. *Given a reduced sub-root system Φ of Ψ_0 , write*

$$\delta_\Phi = \frac{1}{2} \sum_{\alpha \in \Phi \cap \Psi_0^+} \alpha.$$

Let δ'_Φ be the unique dominant weight with respect to Ψ_0^+ in the orbit $W_{\Psi_0}\delta_\Phi$. Denote by $e_{\Psi_0}(\Phi) = |2\delta'_\Phi|^2$ and $e_{\Psi_0} = e_{\Psi_0}(\Psi_0)$.

TABLE 1. Formulas of e_{Ψ_0}

Ψ_0	A_{n-1}	B_n	C_n	D_n	
e_{Ψ_0}	$\frac{(n-1)n(n+1)}{6}$	$\frac{(2n-1)n(2n+1)}{3}$	$\frac{n(n+1)(2n+1)}{3}$	$\frac{n(n-1)(2n-1)}{3}$	
Ψ_0	E_6	E_7	E_8	F_4	G_2
e_{Ψ_0}	156	399	1240	156	28

In the case that Ψ_0 is clear from the context, we simply write $e(\Phi)$ for $e_{\Psi_0}(\Phi)$. Oshima classified sub-root systems Φ of any irreducible root system Ψ_0 up to W_Ψ conjugation. In this section, we give the formulas of $2\delta'_\Phi$ and $e(\Phi)$ for the reduced sub-root systems.

First we have formulas for $e_{\Psi_0} = |2\delta'_{\Psi_0}|^2$ as in Table 1.

Given a reduced sub-root system $\Phi \subset \Psi_0$, denote by

$$\Phi = \bigsqcup_{1 \leq i \leq s} \Phi_i$$

the decomposition of Φ into a disjoint union of irreducible sub-root systems and $\sqrt{k_i}$ the ratio of the length of short roots of Φ_i and Ψ . Then we have

$$e_{\Psi_0}(\Phi) = \sum_{1 \leq i \leq s} k_i e_{\Phi_i}.$$

With this formula we can calculate $e_{\Psi_0}(\Phi)$ quickly from Table 1. On the other hand, from the expression of $2\delta'_\Phi$ into a linear combination of fundamental weights, we can also calculate $e_{\Psi_0}(\Phi) = |2\delta'_\Phi|^2$. This helps on checking whether a formula for $2\delta'_\Phi$ is correct or not. Given a classical irreducible root system Ψ , the calculation of $2\delta'_\Phi$ for sub-root systems Φ of Ψ is easy. We omit it here. For an exceptional irreducible root system Ψ , in Oshima's classification (cf. [Os]), given an abstract root system Φ , sometimes there exist two conjugacy classes of sub-root systems of Ψ isomorphic to Φ . In the following we give representatives of sub-root systems for all the cases when the above ambiguity occurs.

Given $\Psi_0 = E_7$, denote by $\beta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and $\beta' = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. We set

1. $A_5 : \langle \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$,
2. $(A_5)' : \langle \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$,
3. $3A_1 : \langle \alpha_2, \alpha_5, \alpha_7 \rangle$,
4. $(3A_1)' : \langle \alpha_3, \alpha_5, \alpha_7 \rangle$,
5. $4A_1 : \langle \alpha_2, \alpha_3, \alpha_5, \alpha_7 \rangle$,
6. $(4A_1)' : \langle \alpha_2, \alpha_3, \alpha_5, \beta' \rangle$,
7. $A_3 + A_1 : \langle \alpha_2, \alpha_5, \alpha_6, \alpha_7 \rangle$,
8. $(A_3 + A_1)' : \langle \alpha_3, \alpha_5, \alpha_6, \alpha_7 \rangle$,
9. $2A_1 + A_3 : \langle \alpha_2, \alpha_3, \alpha_5, \alpha_6, \alpha_7 \rangle$,
10. $(2A_1 + A_3)' : \langle \beta, \alpha_3, \alpha_5, \alpha_6, \alpha_7 \rangle$,
11. $A_1 + A_5 : \langle \beta, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$,
12. $(A_1 + A_5)' : \langle \beta, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$.

Given $\Psi_0 = E_8$, let $\beta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, $\beta' = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$, and $\beta'' = 2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8$. We set

13. $A_7 : \langle \beta, \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \rangle$,
14. $(A_7)' : \langle \alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \rangle$,
15. $4A_1 : \langle \alpha_2, \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \rangle$,
16. $(4A_1)' : \langle \alpha_2, \alpha_3, \alpha_5, \alpha_8 \rangle$,

TABLE 2. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
D_4	A_1	ω_2	1
D_4	$2A_1^1$	$2\omega_4$	2
D_4	$2A_1'^2$	$2\omega_3$	2
D_4	D_2^3	$2\omega_1$	2
D_4	A_2	$2\omega_2$	4
D_4	$3A_1$	$\omega_1 + \omega_3 + \omega_4$	3
D_4	A_3^4	$2(\omega_2 + \omega_4)$	10
D_4	$A_3'^5$	$2(\omega_2 + \omega_3)$	10
D_4	D_3^6	$2(\omega_2 + \omega_1)$	10
D_4	$4A_1$	$2\omega_2$	4
D_4	D_4	$2(\omega_1 + \omega_2 + \omega_3 + \omega_4)$	28

17. $2A_1 + A_3 : \langle \beta', \alpha_1, \alpha_6, \alpha_7, \alpha_8 \rangle$,
18. $(2A_1 + A_3)' : \langle \alpha_2, \alpha_4, \alpha_3, \alpha_6, \alpha_8 \rangle$,
19. $2A_3 : \langle \beta', \alpha_3, \alpha_1, \alpha_6, \alpha_7, \alpha_8 \rangle$,
20. $(2A_3)' : \langle \alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_7, \alpha_8 \rangle$,
21. $A_1 + A_5 : \langle \beta'', \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \rangle$,
22. $(A_1 + A_5)' : \langle \alpha_1, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \rangle$

Given $\Psi_0 = D_4$, we set

23. $2A_1 : \langle \alpha_1, \alpha_3 \rangle$,
24. $2A_1' : \langle \alpha_1, \alpha_4 \rangle$,
25. $D_2 : \langle \alpha_3, \alpha_4 \rangle$,
26. $A_3 : \langle \alpha_1, \alpha_2, \alpha_3 \rangle$,
27. $A_3' : \langle \alpha_1, \alpha_2, \alpha_4 \rangle$,
28. $D_3 : \langle \alpha_2, \alpha_3, \alpha_4 \rangle$.

8. EQUALITIES AMONG DIMENSION DATA

In this section, we solve Question 5.1. The strategy is as follows. First we reduce it to the case that the root system Ψ is an irreducible root system. In the case that Ψ is of type BC, there is an algebra isomorphism E from the ring of characters to the polynomial ring $\mathbb{Q}[x_0, x_1, \dots]$ defined in [LP]. Moreover, the polynomials in the image are generated by certain polynomials a_n, b_n, c_n, d_n , corresponding to the sub-root systems A_{n-1}, B_n, C_n and D_n , respectively. We are able to get all multiplicative relations among these polynomials $\{a_n, b_n, c_n, d_n \mid n \geq 1\}$. In the case that Ψ is of type B, C or D, the solution in type BC case solves the question in this case as well, with a little more consideration if $\Psi = D_n$. In the case that Ψ is of type A, we show that the characters of non-conjugate sub-root systems have different leading terms. In the case that Ψ is an exceptional irreducible root system, the formulas of the leading terms $\{2\delta'_\Phi \mid \Phi \subset \Psi\}$ have been given in Section 7. We give a case by case discussion of sub-root systems with equal leading terms in their characters.

Theorem 8.1. *Given a root system Ψ and two reduced sub-root systems Φ_1 and Φ_2 , $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$ if and only if there exists $\gamma \in \text{Aut}(\Psi)$ such that*

$$F_{\Phi_i^{(1)}, \text{Aut}(\Psi_i)} = F_{\Phi_i^{(2)}, \text{Aut}(\Psi_i)}$$

TABLE 3. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
E_6	A_1	ω_2	1
E_6	A_2	$2\omega_2$	4
E_6	A_3	$\omega_1 + 2\omega_2 + \omega_6$	10
E_6	A_4	$2\omega_1 + 2\omega_2 + 2\omega_6$	20
E_6	A_5	$2\omega_1 + \omega_2 + \omega_3 + \omega_5 + 2\omega_6$	35
E_6	D_4	$2\omega_2 + 2\omega_4$	28
E_6	D_5	$2\omega_1 + 2\omega_2 + 2\omega_4 + 2\omega_6$	60
E_6	E_6	$2(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6)$	156
E_6	$2A_1$	$\omega_1 + \omega_6$	2
E_6	$3A_1$	ω_4	3
E_6	$4A_1$	$2\omega_2$	4
E_6	$A_2 + A_1$	$\omega_1 + \omega_2 + \omega_6$	5
E_6	$A_2 + 2A_1$	$\omega_3 + \omega_5$	6
E_6	$2A_2$	$2\omega_1 + 2\omega_6$	8
E_6	$2A_2 + A_1$	$\omega_1 + \omega_4 + \omega_6$	9
E_6	$3A_2$	$2\omega_4$	12
E_6	$A_3 + A_1$	$\omega_2 + \omega_3 + \omega_5$	11
E_6	$A_3 + 2A_1$	$2\omega_4$	12
E_6	$A_4 + A_1$	$2\omega_1 + 2\omega_4 + 2\omega_6$	21
E_6	$A_5 + A_1$	$\omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6$	36

for any $1 \leq i \leq m$, where $\Psi = \bigsqcup_{1 \leq i \leq m} \Psi_i$ with each Ψ_i an irreducible root system,

$$\gamma\Phi_1 = \bigsqcup_{1 \leq i \leq m} \Phi_i^{(1)}$$

and

$$\Phi_2 = \bigsqcup_{1 \leq i \leq m} \Phi_i^{(2)}$$

with $\Phi_i^{(1)}, \Phi_i^{(2)} \subset \Psi_i$.

Proof. Since $\text{Aut}(\Psi)$ permutes simple factors of Ψ and it permutes two simple factors if and only if they are isomorphic abstract irreducible root systems, we may assume that $\Psi = m\Psi_0$ and $\text{Aut}(\Psi) = \text{Aut}(\Psi_0)^m \rtimes S_m$ where Ψ_0 is an irreducible root system and $m\Psi_0$ denotes the direct sum of m copies of Ψ_0 .

Denote by $\Lambda = \mathbb{Z}\Psi_0$ the root lattice, by $\mathbb{Q}[\Lambda]$ the character ring and by $U = \mathbb{Q}[\Lambda]^{\text{Aut}(\Psi_0)}$ the invariant characters. Thus U is a \mathbb{Q} vector space with a basis Λ . Write

$$S(U) = \bigoplus_{n \geq 0} S^n(U)$$

for the symmetric tensor algebra over U . It is a polynomial algebra with symmetric tensor product as multiplication. Hence it is a unique factorization domain.

Write $\Phi_j = \bigoplus_{1 \leq i \leq m} \Phi_i^{(j)}$, $j = 1, 2$. In $S(U)$, we have

$$F_{\Phi_j, \text{Aut}(\Psi)} = F_{\Phi_1^{(j)}, \text{Aut}(\Psi_0)} \cdot F_{\Phi_2^{(j)}, \text{Aut}(\Psi_0)} \cdots F_{\Phi_{m-1}^{(j)}, \text{Aut}(\Psi_0)} \cdot F_{\Phi_m^{(j)}, \text{Aut}(\Psi_0)}.$$

As each $F_{\Phi_i^{(j)}, \text{Aut}(\Psi_0)}$ being of degree one and having constant term 1, $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$ if and only if $\{F_{\Phi_i^{(1)}, \text{Aut}(\Psi_0)} \mid 1 \leq i \leq m\}$ and $\{F_{\Phi_i^{(2)}, \text{Aut}(\Psi_0)} \mid 1 \leq i \leq m\}$ differ by a permutation. Therefore the conclusion follows. \square

TABLE 4. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
E ₇	A_1	ω_1	1
E ₇	A_2	$2\omega_1$	4
E ₇	A_3	$2\omega_1 + \omega_6$	10
E ₇	A_4	$2\omega_1 + 2\omega_6$	20
E ₇	A_5 ⁷	$2\omega_1 + 2\omega_6 + 2\omega_7$	35
E ₇	(A_5) ⁸	$\omega_1 + \omega_4 + 2\omega_6$	35
E ₇	A_6	$2\omega_4 + 2\omega_6$	56
E ₇	A_7	$2\omega_1 + 2\omega_4 + 2\omega_6$	84
E ₇	D_4	$2\omega_1 + 2\omega_3$	28
E ₇	D_5	$\omega_1 + 2\omega_3 + 2\omega_6$	60
E ₇	D_6	$2\omega_1 + \omega_2 + \omega_3 + \omega_5 + 2\omega_6 + 2\omega_7$	110
E ₇	E_6	$\omega_1 + \omega_3 + \omega_4 + \omega_6$	156
E ₇	E_7	$2(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7)$	399
E ₇	$2A_1$	ω_6	2
E ₇	$3A_1$ ⁹	$2\omega_7$	3
E ₇	$(3A_1)$ ¹⁰	ω_3	3
E ₇	$4A_1$ ¹¹	$\omega_2 + \omega_7$	4
E ₇	$(4A_1)$ ¹²	$2\omega_1$	4
E ₇	$5A_1$	$\omega_1 + \omega_6$	5
E ₇	$6A_1$	ω_4	6
E ₇	$7A_1$	$2\omega_2$	7
E ₇	$A_2 + A_1$	$\omega_1 + \omega_6$	5
E ₇	$A_2 + 2A_1$	ω_4	6
E ₇	$A_2 + 3A_1$	$2\omega_2$	7
E ₇	$2A_2$	$2\omega_6$	8
E ₇	$2A_2 + A_1$	$\omega_3 + \omega_6$	9
E ₇	$3A_2$	$2\omega_3$	12
E ₇	$A_3 + A_1$ ¹³	$2\omega_1 + 2\omega_7$	11
E ₇	$(A_3 + A_1)$ ¹⁴	$\omega_1 + \omega_4$	11
E ₇	$A_3 + 2A_1$ ¹⁵	$\omega_1 + \omega_5 + \omega_7$	12
E ₇	$(A_3 + 2A_1)$ ¹⁶	$2\omega_3$	12
E ₇	$A_3 + 3A_1$	$\omega_2 + \omega_3 + \omega_7$	13
E ₇	$A_3 + A_2$	$\omega_4 + \omega_6$	14
E ₇	$A_3 + A_2 + A_1$	$2\omega_5$	15
E ₇	$2A_3$	$2\omega_1 + 2\omega_6$	20
E ₇	$2A_3 + A_1$	$\omega_1 + \omega_4 + \omega_6$	21
E ₇	$A_4 + A_1$	$\omega_1 + \omega_4 + \omega_6$	21
E ₇	$A_4 + A_2$	$2\omega_4$	24
E ₇	$A_5 + A_1$ ¹⁷	$\omega_1 + \omega_4 + \omega_6 + 2\omega_7$	36
E ₇	$(A_5 + A_1)$ ¹⁸	$2\omega_3 + 2\omega_6$	36
E ₇	$A_5 + A_2$	$2\omega_4 + 2\omega_7$	39
E ₇	$D_4 + A_1$	$2\omega_1 + \omega_2 + \omega_3 + \omega_7$	29
E ₇	$D_4 + 2A_1$	$2\omega_1 + \omega_4 + \omega_6$	30

Remark 8.2. In the case of $\Psi_0 = \text{BC}_1 = \{\pm e_1, \pm 2e_1\}$,

$$\Lambda = \mathbb{Z}\Psi_0 = \{ne_1 : n \in \mathbb{Z}\}.$$

TABLE 5. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
E_7	$D_4 + 3A_1$	$2\omega_1 + 2\omega_5$	31
E_7	$D_5 + A_1$	$2\omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6$	61
E_7	$D_6 + A_1$	$2\omega_1 + 2\omega_4 + 2\omega_6 + 2\omega_7$	111

TABLE 6. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
E_8	A_1	ω_8	1
E_8	A_2	$2\omega_8$	4
E_8	A_3	$\omega_1 + 2\omega_8$	10
E_8	A_4	$2\omega_1 + 2\omega_8$	14
E_8	A_5	$2\omega_1 + \omega_6 + \omega_8$	35
E_8	A_6	$2\omega_1 + 2\omega_6$	56
E_8	A_7^{19}	$2\omega_1 + 2\omega_6 + 2\omega_8$	84
E_8	$(A_7)^{20}$	$\omega_1 + \omega_4 + \omega_6 + \omega_7$	84
E_8	A_8	$2\omega_4 + 2\omega_7$	120
E_8	D_4	$2\omega_7 + 2\omega_8$	28
E_8	D_5	$2\omega_1 + 2\omega_7 + 2\omega_8$	60
E_8	D_6	$2\omega_1 + \omega_2 + \omega_3 + \omega_7 + 2\omega_8$	110
E_8	D_7	$2\omega_1 + \omega_2 + \omega_3 + \omega_5 + \omega_6 + \omega_8$	182
E_8	D_8	$2\omega_1 + 2\omega_4 + 2\omega_6 + 2\omega_8$	280
E_8	E_6	$\omega_1 + \omega_6 + \omega_7 + \omega_8$	156
E_8	E_7	$2\omega_1 + \omega_2 + \omega_3 + \omega_5 + 2\omega_6 + 2\omega_7 + 2\omega_8$	399
E_8	E_8	$2(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 + \omega_7 + \omega_8)$	1240
E_8	$2A_1$	ω_1	2
E_8	$3A_1$	ω_7	3
E_8	$4A_1^{21}$	$2\omega_8$	4
E_8	$(4A_1)^{22}$	ω_2	4
E_8	$5A_1$	$\omega_1 + \omega_8$	5
E_8	$6A_1$	ω_6	6
E_8	$7A_1$	ω_3	7
E_8	$8A_1$	$2\omega_1$	8
E_8	$A_2 + A_1$	$\omega_1 + \omega_8$	5
E_8	$A_2 + 2A_1$	ω_6	6
E_8	$A_2 + 3A_1$	ω_3	7
E_8	$A_2 + 4A_1$	$2\omega_1$	8
E_8	$2A_2$	$2\omega_1$	8
E_8	$2A_2 + A_1$	$\omega_1 + \omega_7$	9
E_8	$2A_2 + 2A_1$	ω_5	10
E_8	$3A_2$	$2\omega_7$	12
E_8	$3A_2 + A_1$	$\omega_2 + \omega_7$	13
E_8	$4A_2$	$2\omega_2$	16
E_8	$A_3 + A_1$	$\omega_6 + \omega_8$	11

Write $x_n = \frac{[ne_1] + [-ne_1]}{2}$ for any $n \in \mathbb{Z}_{\geq 0}$. In this case $U = \text{span}_{\mathbb{Q}}\{x_n \mid n \geq 0\}$ and

$$S(U) = \mathbb{Q}[x_0, x_1, \dots, x_n, \dots]$$

TABLE 7. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
E_8	$A_3 + 2A_1^{23}$	$2\omega_7$	12
E_8	$(A_3 + 2A_1)^{24}$	$\omega_3 + \omega_8$	12
E_8	$A_3 + 3A_1$	$\omega_2 + \omega_7$	13
E_8	$A_3 + 4A_1$	$\omega_1 + \omega_6$	14
E_8	$A_3 + A_2$	$\omega_1 + \omega_6$	14
E_8	$A_3 + A_2 + A_1$	ω_4	15
E_8	$A_3 + A_2 + 2A_1$	$2\omega_2$	16
E_8	$2A_3^{25}$	$2\omega_1 + 2\omega_8$	20
E_8	$(2A_3)^{26}$	$\omega_1 + \omega_5$	20
E_8	$2A_3 + A_1$	$\omega_1 + \omega_6 + \omega_8$	21
E_8	$2A_3 + 2A_1$	$\omega_4 + \omega_8$	22
E_8	$A_4 + A_1$	$\omega_1 + \omega_6 + \omega_8$	21
E_8	$A_4 + 2A_1$	$\omega_4 + \omega_8$	22
E_8	$A_4 + A_2$	$2\omega_6$	24
E_8	$A_4 + A_2 + A_1$	$\omega_3 + \omega_6$	25
E_8	$A_4 + A_3$	$\omega_4 + \omega_7$	30
E_8	$2A_4$	$2\omega_5$	40
E_8	$A_5 + A_1^{27}$	$2\omega_1 + 2\omega_7$	36
E_8	$(A_5 + A_1)^{28}$	$\omega_1 + \omega_4 + \omega_8$	36
E_8	$A_5 + 2A_1$	$\omega_1 + \omega_5 + \omega_7$	37
E_8	$A_5 + A_2$	$\omega_4 + \omega_6$	39
E_8	$A_5 + A_2 + A_1$	$2\omega_5$	40
E_8	$A_6 + A_1$	$\omega_1 + \omega_4 + \omega_6$	57
E_8	$A_7 + A_1$	$\omega_1 + \omega_4 + \omega_6 + 2\omega_8$	85
E_8	$D_4 + A_1$	$\omega_2 + \omega_7 + 2\omega_8$	29
E_8	$D_4 + 2A_1$	$\omega_1 + \omega_6 + 2\omega_8$	30
E_8	$D_4 + 3A_1$	$\omega_4 + 2\omega_8$	31
E_8	$D_4 + 4A_1$	$2\omega_2 + 2\omega_8$	32
E_8	$D_4 + A_2$	$2\omega_2 + 2\omega_8$	32
E_8	$D_4 + A_3$	$\omega_2 + \omega_3 + \omega_7$	38
E_8	$2D_4$	$2\omega_1 + 2\omega_6$	56
E_8	$D_5 + A_1$	$\omega_1 + \omega_5 + \omega_7 + 2\omega_8$	61
E_8	$D_5 + 2A_1$	$\omega_2 + \omega_3 + \omega_7 + 2\omega_8$	62
E_8	$D_5 + A_2$	$2\omega_5 + 2\omega_8$	64
E_8	$D_5 + A_3$	$\omega_1 + \omega_4 + \omega_6 + \omega_8$	70
E_8	$D_6 + A_1$	$2\omega_1 + \omega_4 + \omega_6 + 2\omega_8$	111
E_8	$D_6 + 2A_1$	$2\omega_1 + 2\omega_5 + 2\omega_8$	112
E_8	$E_6 + A_1$	$\omega_1 + \omega_4 + \omega_6 + 2\omega_7 + 2\omega_8$	157
E_8	$E_6 + A_2$	$\omega_4 + \omega_7 + \omega_8$	160
E_8	$E_7 + A_1$	$2\omega_1 + 2\omega_4 + 2\omega_6 + 2\omega_7 + 2\omega_8$	400

is the usual polynomial algebra. The identification is given by the map E defined in [LP].

Theorem 8.1 reduces Question 5.1 to the case that Ψ is an irreducible root system. We introduce some notation now. Given a root system Ψ , recall that we have defined the integral weight lattice Λ_Ψ (which is a subset of the rational vector space spanned by roots of Ψ) in the “notation and conventions” part.

TABLE 8. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
F_4	A_1^L	ω_1	2
F_4	A_1^S	ω_4	1
F_4	A_2^L	$2\omega_1$	8
F_4	A_2^S	$2\omega_4$	4
F_4	A_3^L	$2\omega_1 + 2\omega_4$	20
F_4	A_3^S	$\omega_1 + 2\omega_4$	10
F_4	D_4^L	$2\omega_1 + 2\omega_2$	56
F_4	D_4^S	$2\omega_3 + 2\omega_4$	28
F_4	B_2	$\omega_1 + 2\omega_4$	10
F_4	B_3	$2\omega_1 + \omega_2 + \omega_4$	35
F_4	C_3	$2\omega_3 + 2\omega_4$	28
F_4	B_4	$2\omega_1 + 2\omega_2 + 2\omega_4$	84
F_4	C_4	$2\omega_1 + 2\omega_3 + 2\omega_4$	60
F_4	F_4	$2\omega_1 + 2\omega_2 + 2\omega_3 + 2\omega_4$	156
F_4	$2A_1^L$	$2\omega_4$	4
F_4	$2A_1^S$	ω_1	2
F_4	$A_1^S + A_1^L$	ω_3	3
F_4	$3A_1^L$	ω_2	6
F_4	$3A_1^S$	ω_3	3
F_4	$A_1^S + 2A_1^L$	$\omega_1 + \omega_4$	5
F_4	$2A_1^S + A_1^L$	$2\omega_4$	4
F_4	$4A_1^L$	$2\omega_1$	8
F_4	$4A_1^S$	$2\omega_4$	4
F_4	$2A_1^S + 2A_1^L$	ω_2	6
F_4	$A_2^L + A_1^S$	$\omega_1 + \omega_3$	9
F_4	$A_2^S + A_1^L$	ω_2	6
F_4	$A_2^S + A_2^L$	$2\omega_3$	12
F_4	$B_2 + A_1^L$	$2\omega_3$	12
F_4	$B_2 + A_1^S$	$\omega_2 + \omega_4$	11
F_4	$B_2 + 2A_1^L$	$\omega_1 + \omega_4$	14
F_4	$B_2 + 2A_1^S$	$2\omega_3$	12
F_4	$2B_2$	$2\omega_1 + 2\omega_4$	20
F_4	$A_3^S + A_1^L$	$2\omega_3$	12
F_4	$A_3^L + A_1^S$	$\omega_1 + \omega_2 + \omega_4$	21
F_4	$C_3 + A_1^L$	$\omega_1 + \omega_2 + 2\omega_4$	30
F_4	$B_3 + A_1^S$	$2\omega_1 + 2\omega_3$	36

TABLE 9. Formulas of $2\delta'_\Phi$

Ψ_0	Φ	$2\delta'_\Phi$	$e(\Phi)$
G_2	A_1^L	ω_1	3
G_2	A_1^S	ω_2	1
G_2	A_2^L	$2\omega_1$	12
G_2	A_2^S	$2\omega_2$	4
G_2	G_2	$2(\omega_1 + \omega_2)$	28
G_2	$A_1^S + A_1^L$	$2\omega_2$	4

Definition 8.3. *Given an integral weight λ , let*

$$\chi_{\lambda, \Psi}^* = \frac{1}{|W_\Psi|} \sum_{\gamma \in W_\Psi} [\gamma\lambda].$$

In the case that the root system Ψ is clear from the context, we simply write χ_λ^* for $\chi_{\lambda, \Psi}^*$. As discussed in Section 3, the characters $\{\chi_\lambda^* \mid \lambda \in \Lambda_\Psi^+\}$ is a basis of the vector space $\mathbb{Q}[\Lambda_\Psi]^{W_{\Psi_0}}$, where Λ_Ψ^+ is the set of dominant weights in Λ_Ψ (with respect to a positive system determined by a chosen order).

Combining Theorems 8.13 and 8.14, we get the following theorem. It answers Question 5.1 completely in the case that Ψ is an irreducible root system.

Theorem 8.4. *Given an irreducible root system Ψ , if there exist two non-conjugate reduced sub-root systems $\Phi_1, \Phi_2 \subset \Psi$ with $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$, then $\Psi \cong C_n, BC_n$ or F_4 .*

In the case that $\Psi = C_n$ or BC_n , $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$ if and only if

$$b_m(\Phi_1) - b_m(\Phi_2) = a_{2m}(\Phi_1) - a_{2m}(\Phi_2) = 0$$

and

$$a_{2m-1}(\Phi_1) - a_{2m-1}(\Phi_2) = c_{m-1}(\Phi_2) - c_{m-1}(\Phi_1) = d_m(\Phi_2) - d_m(\Phi_1)$$

for any $m \geq 1$. Here $a_m(\Phi_i)$, $b_m(\Phi_i)$, $c_m(\Phi_i)$, $d_m(\Phi_i)$ is the number of simple factors of $\Phi_i \subset \Psi \subset BC_n$ isomorphic to A_{m-1} , B_m , C_m or D_m , respectively.

In the case that $\Psi = F_4$, $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$ if and only if

$$\begin{aligned} \Phi_1 &\sim \Phi_2, \\ \{\Phi_1, \Phi_2\} &\sim \{A_2^S, A_1^L + 2A_1^S\} \end{aligned}$$

or

$$\{\Phi_1, \Phi_2\} \sim \{A_1^L + A_2^S, 2A_1^L + 2A_1^S\}.$$

8.1. Classical irreducible root systems. As in Section 3 of [LP], let

$$\begin{aligned} \mathbb{Z}^n &:= \mathbb{Z} BC_n = \Lambda_{BC_n} = \text{span}_{\mathbb{Z}}\{e_1, e_2, \dots, e_n\}, \\ W_n &:= \text{Aut}(BC_n) = W_{BC_n} = \{\pm 1\}^n \rtimes S_n, \\ \mathbb{Z}_n &:= \mathbb{Q}[\mathbb{Z}^n], \\ Y_n &:= \mathbb{Z}_n^{W_n}. \end{aligned}$$

For $m \leq n$, the injection

$$\mathbb{Z}^m \hookrightarrow \mathbb{Z}^n : (a_1, \dots, a_m) \mapsto (a_1, \dots, a_m, 0, \dots, 0)$$

extends to an injection $i_{m,n} : \mathbb{Z}_m \hookrightarrow \mathbb{Z}_n$. Define $\phi_{m,n} : \mathbb{Z}_m \rightarrow \mathbb{Z}_n$ by

$$\phi_{m,n}(z) = \frac{1}{|W_n|} \sum_{w \in W_n} w(i_{m,n}(z)).$$

Thus $\phi_{m,n}\phi_{k,m} = \phi_{k,n}$ for any $k \leq m \leq n$ and the image of $\phi_{m,n}$ lies in Y_n . Hence $\{Y_m : \phi_{m,n}\}$ forms a direct system and we define

$$Y = \varinjlim Y_n.$$

Define the map $j_n : \mathbb{Z}_n \rightarrow Y$ by composing $\phi_{n,p}$ with the injection $Y_p \hookrightarrow Y$. The isomorphism $\mathbb{Z}^m \oplus \mathbb{Z}^n \xrightarrow{\sim} \mathbb{Z}^{m+n}$ gives a canonical isomorphism $M : \mathbb{Z}_m \otimes_{\mathbb{Q}} \mathbb{Z}_n \xrightarrow{\sim} \mathbb{Z}_{m+n}$. Given two elements of Y represented by $y \in Y_m$ and $y' \in Y_n$ we define

$$yy' = j_{m+n}(M(y \otimes y')).$$

This product is independent of the choice of m and n and makes Y a commutative associative algebra. The monomials $[e_1]^{a_1} \cdots [e_n]^{a_n}$ ($a_1, a_2, \dots, a_n \in \mathbb{Z}$) form a \mathbb{Q} basis of \mathbb{Z}_n , where $[e_i]^{a_i} = [a_i e_i] \in \mathbb{Z}_1$ is a linear character. Hence Y has a \mathbb{Q} basis

$$e(a_1, a_2, \dots, a_n) = j_n([e_1]^{a_1} \cdots [e_n]^{a_n})$$

indexed by $n \geq 0$ and $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Mapping $e(a_1, a_2, \dots, a_n)$ to $x_{a_1} x_{a_2} \cdots x_{a_n}$, we get a \mathbb{Q} linear map

$$E : Y \longrightarrow \mathbb{Q}[x_0, x_1, \dots, x_n, \dots].$$

Lemma 8.5. ([LP] Page 390) *The above map E is an algebra isomorphism.*

Proof. The bijectivity is clear. It is an isomorphism follows from the equality

$$e(a_1, a_2, \dots, a_m) e(b_1, b_2, \dots, b_n) = e(c_1, c_2, \dots, c_{m+n}),$$

where $(c_1, c_2, \dots, c_{m+n})$ is the re-permutation of $\{a_1, \dots, a_m, b_1, \dots, b_n\}$ in decreasing order. This equality can be proved by a combinatorial calculation. \square

Definition 8.6. Write a_n, b_n, c_n, d_n for the image of $j_n(F_{\Phi, W_n})$ under E for $\Phi = A_{n-1}, B_n, C_n$ or D_n , respectively.

Observe that a_n, b_n, c_n, d_n are homogeneous polynomials of degree n with integer coefficients and a term x_0^n .

Remark 8.7. Here, E maps linear characters to homogeneous polynomials. Letting $x_0 = 1$, then the definition here becomes that in [LP]. In this terminology, each of a_n, b_n, c_n, d_n has integer coefficients and constant term 1.

Note that our b_n, c_n, d_n are b'_n, c'_n, d'_n in [LP]. For any $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $\lambda = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$, one sees that

$$\chi_{\lambda, W_n}^* = [e_1]^{a_1} \cdots [e_n]^{a_n}$$

in Y . Thus $E(\chi_{\lambda, W_n}^*) = x_{a_1} x_{a_2} \cdots x_{a_n}$.

For small n , we have $a_1 = d_1 = x_0, b_1 = x_0 - x_1, c_1 = x_0 - x_2$,

$$a_2 = x_0^2 - x_1^2,$$

$$b_2 = x_0^2 - x_0 x_1 - x_0 x_3 + x_1 x_3 - x_1^2 + 2x_1 x_2 - x_2^2,$$

$$c_2 = x_0^2 - x_0 x_2 - x_0 x_4 + x_2 x_4 - x_1^2 + 2x_1 x_3 - x_3^2,$$

$$d_2 = x_0^2 - 2x_1^2 + x_0 x_2$$

and $a_3 = x_0^3 - 2x_0 x_1^2 + 2x_1^2 x_2 - x_0 x_2^2$.

For the convenience in writing notations, we define $a_0 = b_0 = c_0 = d_0 = 1$ and $c_{-1} = x_0^{-1}$.

Proposition 8.8. ([LP], Page 390)

- (1) We have $c_n, d_{n+1} \in \mathbb{Q}[x_1, x_2, \dots, x_{2n}] - \mathbb{Q}[x_1, x_2, \dots, x_{2n-1}]$,
 $b_n \in \mathbb{Q}[x_1, x_2, \dots, x_{2n-1}] - \mathbb{Q}[x_1, x_2, \dots, x_{2n-2}]$.
- (2) Each of b_n, c_n, d_{n+1} is a prime in $\mathbb{Q}[x_1, x_2, \dots]$ and any two of them are different.
- (3) Each of the subsets $\{b_1, \dots, b_n, c_1, \dots, c_n\}$, $\{b_1, \dots, b_n, d_2, \dots, d_{n+1}\}$,
 $\{c_1, \dots, c_n, d_2, \dots, d_{n+1}\}$ is algebraically independent.

Given $f \in \mathbb{Q}[x_0, x_1, \dots]$, let

$$\sigma(f)(x_0, x_1, \dots, x_{2n}, x_{2n+1}, \dots) = f(x_0, -x_1, \dots, x_{2n}, -x_{2n+1}, \dots).$$

Then σ is an involutive automorphism of $\mathbb{Q}[x_0, x_1, \dots]$.

Proposition 8.9. We have $\sigma(a_n) = a_n$, $\sigma(c_n) = c_n$, $\sigma(d_n) = d_n$ for any n and but $\sigma(b_n) \neq b_n$ when $n \geq 1$.

Proof. This follows from the formula

$$F_{\Phi, W_n} = \sum_{w \in W_\Phi} \epsilon(w) \chi_{\delta_\Phi - w\delta_\Phi, W_n}^*$$

and the expression of δ_Φ for $\Phi = A_{n-1}, B_n, C_n, D_n$. \square

Definition 8.10. Define $b'_n = \sigma(b_n)$.

Proposition 8.11. For any $n \geq 1$, we have $a_{2n} = b_n b'_n$ and $a_{2n+1} = c_n d_{n+1}$.

Proof. Given $n \geq 1$, define the matrices $A_n = (x_{|i-j|})_{n \times n}$,

$$B_n = (x_{|i-j|} - x_{i+j-1})_{n \times n},$$

$$C_n = (x_{|i-j|} - x_{i+j})_{n \times n},$$

$$B'_n = (x_{|i-j|} + x_{i+j-1})_{n \times n},$$

$$D_n = (x_{|i-j|} + x_{i+j-2})_{n \times n},$$

$$D'_n = (a_{i,j})_{n \times n},$$

where $a_{i,j} = x_{|i-j|} + x_{i+j-2}$ if $i, j \geq 2$, $a_{1,j} = a_{j,1} = \sqrt{2}x_{j-1}$, $a_{1,1} = 1$. Then we have the following equalities:

$$(3) \quad a_n = \det A_n,$$

$$(4) \quad b_n = \det B_n,$$

$$(5) \quad b'_n = \det B'_n,$$

$$(6) \quad c_n = \det C_n,$$

$$(7) \quad d_n = \frac{1}{2} \det D_n = \det D'_n.$$

We prove the equality (3). The others can be proved similarly. Recall that

$$F_{A_{n-1}, W_n} = \sum_{w \in S_n} \epsilon(w) \chi_{\delta - w\delta, W_n}^*,$$

where

$$\begin{aligned} \delta &= \frac{(n-1)e_1 + (n-3)e_2 + \cdots + (1-n)e_n}{2} \\ &= (ne_1 + (n-1)e_2 + \cdots + e_n) - \frac{n+1}{2}(e_1 + e_2 + \cdots + e_n). \end{aligned}$$

Let $\delta' = ne_1 + (n-1)e_2 + \cdots + e_n$. Then

$$F_{A_{n-1}, W_n} = \sum_{w \in S_n} \epsilon(w) \chi_{\delta' - w\delta', W_n}^*.$$

Expanding $\det A_n = \det(x_{|i-j|})_{n \times n}$ into the sum of terms according to permutations, a term corresponding to a permutation $w \in S_n$ is equal to the polynomial $E(j_n(\chi_{\delta' - w\delta', W_n}^*))$. Summing up all terms, we get $a_n = \det A_n$.

Let

$$\begin{aligned} L_n &= \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix}, \\ J_{2n} &= \begin{pmatrix} \frac{1}{\sqrt{2}}I_n & -\frac{1}{\sqrt{2}}L_n \\ \frac{1}{\sqrt{2}}L_n & \frac{1}{\sqrt{2}}I_n \end{pmatrix} \end{aligned}$$

and

$$J_{2n+1} = \begin{pmatrix} \frac{1}{\sqrt{2}}I_n & \vdots & -\frac{1}{\sqrt{2}}L_n \\ \cdots & 1 & \vdots \\ \frac{1}{\sqrt{2}}L_n & \vdots & \frac{1}{\sqrt{2}}I_n \end{pmatrix}.$$

By matrix calculation, we have

$$J_{2n}A_{2n}J_{2n}^{-1} = \begin{pmatrix} B_n & \\ & B'_n \end{pmatrix},$$

$$J_{2n+1}A_{2n+1}J_{2n+1}^{-1} = \begin{pmatrix} C_n & \\ & D'_{n+1} \end{pmatrix}.$$

Taking determinants, we get $a_{2n} = b_nb'_n$ and $a_{2n+1} = c_nd_{n+1}$. \square

Corollary 8.12. *Any multiplicative relation among $\{a_{n+1}, b_n, c_n, d_{n+1} | n \geq 1\}$ is generated by $\{a_{2n+1} = c_nd_{n+1} : n \geq 1\}$.*

Proof. This follows from Proposition 8.8 and Proposition 8.11. \square

Theorem 8.13. *Given a classical irreducible root system Ψ , if there exists non-conjugate sub-root systems Φ_1 and Φ_2 of Ψ such that $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$, then $\Psi \cong C_n$ or BC_n .*

In the case that $\Psi = C_n$ or BC_n , $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$ if and only if

$$\forall m \leq n, b_m(\Phi_1) - b_m(\Phi_2) = a_{2m}(\Phi_1) - a_{2m}(\Phi_2) = 0$$

$$\text{and } a_{2m+1}(\Phi_1) - a_{2m+1}(\Phi_2) = c_m(\Phi_2) - c_m(\Phi_1) = d_{m+1}(\Phi_2) - d_{m+1}(\Phi_1).$$

Here $a_m(\Phi_i), b_m(\Phi_i), c_m(\Phi_i), d_m(\Phi_i)$ is the number of simple factors of $\Phi_i \subset BC_n$ isomorphic to A_{m-1}, B_m, C_m, D_m respectively.

Note that, we embed A_{m-1}, B_m, C_m, D_m into BC_n ($n \geq m$) in the standard way. This means that the sub-root systems A_1, B_1, C_1 are non-isomorphic to each other, as well as each of the pairs $(B_2, C_2), (D_2, A_1 \sqcup A_1)$ and (A_3, D_3) .

Proof of Theorem 8.13. In the case of $\Psi = C_n$ or BC_n , since $j_n : Y_n \rightarrow Y$ is an injection, $F_{\Phi_1, W_n} = F_{\Phi_2, W_n}$ if and only if $E(F_{\Phi_1, W_n}) = E(F_{\Phi_2, W_n})$, i.e.

$$\prod_{1 \leq i \leq n} (a_i^{r_i^{(1)}} b_i^{s_i^{(1)}} c_i^{u_i^{(1)}} d_i^{v_i^{(1)}}) = \prod_{1 \leq i \leq n} (a_i^{r_i^{(2)}} b_i^{s_i^{(2)}} c_i^{u_i^{(2)}} d_i^{v_i^{(2)}}).$$

Here we write $r_m^{(j)} = a_m(\Phi_j), s_m^{(j)} = b_m(\Phi_j), u_m^{(j)} = c_m(\Phi_j), v_m^{(j)} = d_m(\Phi_j)$. Therefore the conclusion follows from Corollary 8.12.

In the case that $\Psi = B_n$ ($n \geq 1$) or D_n ($n \geq 5$), $\text{Aut}(\Psi_0) = W_n$. Since any C_k is not contained in B_n or D_n , the conclusion follows from the conclusion for BC_n case.

In the case that $\Psi = D_4$, only the characters of the non-conjugate sub-root systems $A_2, 4A_1$ have the equal leading term, which is $\chi_{2\omega_2}$. We have

$$F_{A_2, W_\Psi} = 1 - 2\chi_{\omega_2}^* + 2\chi_{\omega_1 + \omega_3 + \omega_4}^* - \chi_{2\omega_2}^*$$

and

$$F_{4A_1, W_\Psi} = 1 - 4\chi_{\omega_2}^* + 2(\chi_{2\omega_1}^* + \chi_{2\omega_3}^* + \chi_{2\omega_4}^*) - 4\chi_{\omega_1 + \omega_3 + \omega_4}^* + \chi_{2\omega_2}^*.$$

Thus $F_{A_2, W_\Psi} \neq F_{4A_1, W_\Psi}$. Therefore the conclusion follows.

In the case that $\Psi = A_{n-1}$, $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$ implies that $2\delta_{\Phi_1} \sim_{\text{Aut}(\Psi)} 2\delta_{\Phi_2}$. And the latter implies that $\Phi_1 \sim_{\text{Aut}(\Psi)} \Phi_2$. Therefore the conclusion follows. \square

8.2. Exceptional irreducible root systems. In Section 7, we give the formulas of $2\delta'_\Phi$ and their modulus squares for reduced sub-root systems of the exceptional simple root systems $\Psi = E_6, E_7, E_8, F_4$ or G_2 . Using these formulas, in this section we classify non-conjugate sub-root systems Φ_1, Φ_2 of any exceptional irreducible root system Ψ such that

$$F_{\Phi_1, W_\Psi} = F_{\Phi_2, W_\Psi}.$$

Theorem 8.14. *Given an irreducible root system $\Psi = E_6, E_7, E_8, F_4$ or G_2 , if there exists non-conjugate reduced sub-root systems Φ_1 and Φ_2 of Ψ such that $F_{\Phi_1, \text{Aut}(\Psi)} = F_{\Phi_2, \text{Aut}(\Psi)}$, then $\Psi \cong F_4$.*

In the case that $\Psi = F_4$, $F_{\Phi_1, \Psi} = F_{\Phi_2, \Psi}$ and $\Phi_1 \not\sim \Phi_2$ if and only if

$$\{\Phi_1, \Phi_2\} \sim \{A_2^S, A_1^L + 2A_1^S\} \text{ or } \{A_1^L + A_2^S, 2A_1^L + 2A_1^S\}.$$

Proof. In the case that $\Psi = E_6$, among the dominant integral weights appearing in $\{2\delta'_\Phi \mid \Phi \subset E_6\}$, those weights that appearing more than once include $\{2\omega_2, 2\omega_4\}$ and the sub-root systems Φ with $2\delta_\Phi$ conjugate to them are

- (1) $2\omega_2$: $A_2, 4A_1^S$, appears 2 times.
- (2) $2\omega_4$: $3A_2, A_3 + 2A_1$, appears 2 times.

The coefficients of $\chi_{\omega_2}^*$ in $F_{A_2, \text{Aut}(E_6)}, F_{4A_1^S, \text{Aut}(E_6)}$ are different and the coefficients of $\chi_{\omega_2}^*$ in $F_{3A_2, \text{Aut}(E_6)}, F_{A_3+2A_1, \text{Aut}(E_6)}$ are also different. Therefore the conclusion in the E_6 case follows.

In the case that $\Psi = E_7$, among the dominant integral weights appearing in $\{2\delta'_\Phi \mid \Phi \subset E_7\}$, those weights that appearing more than once include

$$\{2\omega_1, \omega_1 + \omega_6, \omega_4, 2\omega_2, 2\omega_3, 2\omega_1 + 2\omega_6, \omega_1 + \omega_4 + \omega_6\}$$

and the sub-root systems Φ with $2\delta_\Phi$ conjugate to them are

- (1) $2\omega_1$: $A_2, 4A_1'$, appears two times.
- (2) $\omega_1 + \omega_6$: $A_2 + A_1, 5A_1$, appears two times.
- (3) ω_4 : $A_2 + 2A_1, 6A_1$, appears two times.
- (4) $2\omega_2$: $A_2 + 3A_1, 7A_1$, appears two times.
- (5) $2\omega_3$: $3A_2, A_3 + 2A_1$, appears two times.
- (6) $2\omega_1 + 2\omega_6$: $A_4, 2A_3$, appears two times.
- (7) $\omega_1 + \omega_4 + \omega_6$: $A_4 + A_1, 2A_3 + A_1$, appears two times.

Given a weight λ , there are at most two conjugacy classes of sub-root systems Φ_1, Φ_2 of E_7 such that $2\delta'_\Phi = \lambda$. For each of such λ , the numbers of simple roots in Φ_1, Φ_2 are non-equal. Thus the coefficients of $\chi_{\omega_1}^*$ in $F_{\Phi_1, W_{E_7}}$ and $F_{\Phi_2, W_{E_7}}$ are different. Therefore the conclusion in the E_7 case follows.

In the case that $\Psi = E_8$, among the dominant integral weights appearing in $\{2\delta'_\Phi \mid \Phi \subset E_8\}$, those weights that appearing more than once include

$$\{2\omega_8, \omega_1 + \omega_8, \omega_6, \omega_3, 2\omega_1, 2\omega_7, \omega_2 + \omega_7, \omega_1 + \omega_6, 2\omega_2, 2\omega_1 + 2\omega_8, \omega_1 + \omega_6 + \omega_8, \\ \omega_4 + \omega_8, 2\omega_2 + 2\omega_8, 2\omega_5, 2\omega_1 + 2\omega_6\}$$

and the sub-root systems Φ with $2\delta_\Phi$ conjugate to them are

- (1) $2\omega_8$: $A_2, 4A_1'$, appears two times.
- (2) $\omega_1 + \omega_8$: $A_2 + A_1, 5A_1$, appears two times.
- (3) ω_6 : $A_2 + 2A_1, 6A_1$, appears two times.
- (4) ω_3 : $A_2 + 3A_1, 7A_1$, appears two times.
- (5) $2\omega_1$: $2A_2, A_2 + 4A_1, 8A_1$, appears three times.
- (6) $2\omega_7$: $A_3 + 2A_1, 3A_2$, appears two times.
- (7) $\omega_2 + \omega_7$: $A_3 + 3A_1, 3A_2 + A_1$, appears two times.

- (8) $\omega_1 + \omega_6$: $A_3 + 4A_1$, $A_3 + A_2$, appears two times.
- (9) $2\omega_2$: $A_3 + A_2 + 2A_1$, $4A_2$, appears two times.
- (10) $2\omega_1 + 2\omega_8$: A_4 , $2A_3$, appears two times.
- (11) $\omega_1 + \omega_6 + \omega_8$: $A_4 + A_1$, $2A_3 + A_1$, appears two times.
- (12) $\omega_4 + \omega_8$: $A_4 + 2A_1$, $2A_3 + 2A_1$, appears two times.
- (13) $2\omega_2 + 2\omega_8$: $D_4 + A_2$, $D_4 + 4A_1$, appears two times.
- (14) $2\omega_5$: $A_5 + A_2 + A_1$, $2A_4$, appears two times.
- (15) $2\omega_1 + 2\omega_6$: A_6 , $2D_4$, appears two times.

One sees: for any two non-conjugate sub-root systems $\Phi_1, \Phi_2 \subset E_8$ with $2\delta'_{\Phi_1} = 2\delta'_{\Phi_2}$, the numbers of simple roots of Φ_1, Φ_2 are non-equal, so the coefficients of $\chi_{\omega_1}^*$ in $F_{\Phi_1, W_{E_8}}, F_{\Phi_2, W_{E_8}}$ are different. Thus $F_{\Phi_1, W_{E_8}} \neq F_{\Phi_2, W_{E_8}}$. Therefore the conclusion in the E_8 case follows.

In the case that $\Psi = F_4$, among the dominant integral weights appearing in $\{2\delta'_\Phi : \Phi \subset F_4\}$, those weights that appearing more than once include

$$\{\omega_1, \omega_3, 2\omega_4, \omega_2, 2\omega_1, \omega_1 + 2\omega_4, 2\omega_3, 2\omega_1 + 2\omega_4, 2\omega_3 + 2\omega_4\}$$

and the sub-root systems Φ with $2\delta_\Phi$ conjugate to them are

- (1) ω_1 : $A_1^L, 2A_1^S$, appears 2 times.
- (2) ω_3 : $A_1^L + A_1^S, 3A_1^S$, appears 2 times.
- (3) $2\omega_4$: $A_2^S, 2A_1^L, A_1^L + 2A_1^S, 4A_1^S$, appears 4 times.
- (4) ω_2 : $3A_1^L, 2A_1^L + 2A_1^S, A_1^L + A_2^S$, appears 3 times.
- (5) $2\omega_1$: $A_2^L, 4A_1^L$, appears 2 times.
- (6) $\omega_1 + 2\omega_4$: A_3^S, B_2 , appears 2 times.
- (7) $2\omega_3$: $A_2^L + A_2^S, A_1^L + B_2, 2A_1^S + B_2, A_1^L + A_3^S$, appears 4 times.
- (8) $2\omega_1 + 2\omega_4$: $A_3^L, 2B_2$, appears 2 times.
- (9) $2\omega_3 + 2\omega_4$: D_4^S, C_3 , appears 2 times.

The non-conjugate pairs of sub-root system $\Phi_1, \Phi_2 \subset F_4$ with conjugate leading terms $2\delta'_{\Phi_i}$ and the same number of short simple roots are $(A_2^S, A_1^L + 2A_1^S), (2A_1^L + 2A_1^S, A_1^L + A_2^S), (A_2^L, 4A_1^L), (2A_1^S + B_2, A_1^L + A_3^S)$. The coefficients of shortest terms in $F_{A_2^L, F_4}, F_{4A_1^L, F_4}$ are non-equal, and $F_{2A_1^S + B_2}$ is not equal to $F_{A_1^L + A_3^S}$ since

$$\begin{aligned} F_{2A_1^S + B_2, W_{F_4}} &= 1 - 3\chi_{\omega_4}^* + 2\chi_{\omega_1}^* + \chi_{\omega_3}^* - \chi_{2\omega_4}^* + 2\chi_{\omega_1 + \omega_4}^* - 4\chi_{\omega_2}^* + 2\chi_{\omega_3 + \omega_4}^* - \\ &\quad \chi_{2\omega_1}^* + 2\chi_{\omega_1 + \omega_3}^* - \chi_{3\omega_4}^* + 2\chi_{\omega_1 + 2\omega_4}^* - 3\chi_{\omega_2 + \omega_4}^* + \chi_{2\omega_3}^* \end{aligned}$$

and

$$\begin{aligned} F_{A_1^L + A_3^S, W_{F_4}} &= 1 - 3\chi_{\omega_4}^* + 7\chi_{\omega_3}^* - 3\chi_{2\omega_4}^* - 6\chi_{\omega_1 + \omega_4}^* + 6\chi_{\omega_3 + \omega_4}^* + 3\chi_{2\omega_1}^* - \\ &\quad 6\chi_{\omega_1 + \omega_3}^* - \chi_{3\omega_4}^* + 3\chi_{\omega_2 + \omega_4}^* - \chi_{2\omega_3}^*. \end{aligned}$$

Calculation shows that

$$F_{A_2^S, W_{F_4}} = F_{A_1^L + 2A_1^S, W_{F_4}} = 1 - 2\chi_{\omega_4}^* + 2\chi_{\omega_3}^* - \chi_{2\omega_4}^*$$

and

$$F_{A_1^L + A_2^S, W_{F_4}} = F_{2A_1^L + 2A_1^S, W_{F_4}} = 1 - 2\chi_{\omega_4}^* - \chi_{\omega_1}^* + 4\chi_{\omega_3}^* - \chi_{2\omega_4}^* - 2\chi_{\omega_1 + \omega_4}^* + \chi_{\omega_2}^*.$$

Therefore the conclusion in the F_4 case follows.

In the case that $\Psi = G_2$, the only non-conjugate pair (Φ_1, Φ_2) of sub-root systems such that $2\delta'_{\Phi_1} = 2\delta'_{\Phi_2}$ is $(A_2^S, A_1^L + A_1^S)$. The numbers of short simple roots of A_2^S and $A_1^L + A_1^S$ are different, so the coefficients of $\chi_{\omega_2}^*$ in $F_{A_2^S, W_{G_2}}, F_{A_1^L + A_1^S, W_{G_2}}$ are non-equal. Thus $F_{\Phi_1, W_{G_2}} \neq F_{\Phi_2, W_{G_2}}$. Therefore the conclusion in the G_2 case follows. \square

9. LINEAR RELATIONS AMONG DIMENSION DATA

In this section, we solve Question 5.2.

First, once we know all linear relations among $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$, we also know all linear relations among $\{F_{\Phi, W} \mid \Phi \subset \Psi\}$ for any finite group W between W_Ψ and $\text{Aut}(\Psi)$. So we just need to consider the linear relations among $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$.

If Ψ is not an irreducible root system, let

$$\Psi = \bigsqcup_{1 \leq i \leq s} \Psi_i$$

be the decomposition of Ψ into a direct sum of simple root systems. For a reduced sub-root system Φ of Ψ , Φ can be written as $\Phi = \bigsqcup_{1 \leq i \leq s} \Phi_i$, where $\Phi_i \subset \Psi_i$ for any $1 \leq i \leq s$. Thus we have

$$F_{\Phi, W_\Psi} = F_{\Phi_1, W_{\Psi_1}} \otimes \cdots \otimes F_{\Phi_s, W_{\Psi_s}}.$$

From this, we see that linear relations among $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$ arise from linear relations among $\{F_{\Phi_i, W_{\Psi_i}} \mid \Phi_i \subset \Psi_i, 1 \leq i \leq s\}$. Hence it is sufficient to consider $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$ for reduced sub-root systems $\{\Phi \mid \Phi \subset \Psi\}$ of an irreducible root system Ψ .

Remark 9.1. *The passing from $\{F_{\Phi, W} \mid \Phi \subset \Psi\}$ to $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$ is like that: any linear relation among the former is also a linear relation among the latter; and any linear relation among the latter gives a linear relation among the former after the W -averaging process, that is to replace a character $F_{\Phi, W_\Psi} \in \mathbb{Q}[\Lambda]$ by*

$$F_{\Phi, W} = \frac{1}{|W|} \sum_{\gamma \in W} \gamma F_{\Phi, W_\Psi}.$$

But the relation between the two sets of linear relations might not be described explicitly, there are at least two reasons for this. The first reason is the number of distinct characters in $\{\gamma F_{\Phi, W_\Psi} : \gamma \in W/W_\Psi\}$ may vary; the second and the more serious reason is different linear relations among $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$ may give the same linear relation among $\{F_{\Phi, W} \mid \Phi \subset \Psi\}$ after the W -averaging process.

The passing from $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$ to $\{F_{\Phi_i, W_{\Psi_i}} \mid \Phi_i \subset \Psi_i\}$ is reasonably well, since by Linear Algebra all linear relations among the former can be explicitly expressed in terms of linear relations among the latter.

Remark 9.2. *Another way of getting all linear relations among $\{F_{\Phi, W} \mid \Phi \subset \Psi\}$ is to express each $F_{\Phi, W}$ in terms of $\{F_{\Phi_i, W_{\Psi_i}}\}$ by the tensor operation and the W -averaging operation. Starting from all linear relations among $\{F_{\Phi_i, W_{\Psi_i}} : \Phi_i \subset \Psi_i\}$, in this way we can get all linear relations among $\{F_{\Phi, W} \mid \Phi \subset \Psi\}$.*

For two reduced sub-root systems $\Phi_1, \Phi_2 \subset \Psi$ with $\Phi_2 \not\sim_W \Phi_1$, the above way is useful for checking if $F_{\Phi_1, W} = F_{\Phi_2, W}$ or not. We express $F_{\Phi_1, W} - F_{\Phi_2, W}$ in terms of $\{F_{\Phi_i, W_{\Psi_i}}\}$ by tensor operation, addition and subtraction. Then the linear relations among $\{F_{\Phi_{i,j}, W_{\Psi_i}} : \Phi_{i,j} \subset \Psi_i\}$ tells us if such an expression is 0 or not. For example, in the case that $\Psi = m\Psi_0$ and $W = (W_{\Psi_0})^m \rtimes A_m$, if $\Phi_1, \dots, \Phi_m \subset \Psi_0$ are reduced sub-root systems such that $F_{\Phi_1, W_{\Psi_0}}, \dots, F_{\Phi_m, W_{\Psi_0}}$ are linearly dependent, then for

$$\Phi = \Phi_1 \sqcup \Phi_2 \sqcup \cdots \sqcup \Phi_m$$

and

$$\Phi = \Phi_2 \sqcup \Phi_1 \sqcup \Phi_3 \sqcup \Phi_4 \sqcup \cdots \sqcup \Phi_m,$$

we have $F_{\Phi_1, W} = F_{\Phi_2, W}$.

9.1. Algebraic relations among $\{a_n, b_n, b'_n, c_n, d_n \mid n \geq 1\}$. The following proposition can be proven in a similar way as the proof of Proposition 8.11. The method is: after showing equalities (3)-(7), we are led to prove some identities for determinants.

Proposition 9.3. *For any $n \geq 0$, we have $a_{2n} = b_n b'_n$, $a_{2n+1} = c_n d_{n+1}$, $2a_{2n} = c_n d_n + c_{n-1} d_{n+1}$ and $2a_{2n+1} = b_n b'_{n+1} + b'_n b_{n+1}$.*

Proof. The equalities

$$(8) \quad a_{2n} = b_n b'_n$$

and

$$(9) \quad a_{2n+1} = c_n d_{n+1}$$

are proven in Proposition 8.11. We prove

$$(10) \quad 2a_{2n} = c_n d_n + c_{n-1} d_{n+1}$$

and

$$(11) \quad 2a_{2n+1} = b_n b'_{n+1} + b'_n b_{n+1}$$

here. In the proof of Proposition 8.11, we have introduced the matrices A_n, B_n, C_n, D_n, D'_n and expressed their determinants in terms of the polynomials a_n, b_n, c_n, d_n .

Let

$$A'_{2n} = \begin{pmatrix} x_0 & x_1 & \dots & x_{2n-2} & x_{2n-1} \\ x_1 & x_0 & \dots & x_{2n-3} & x_{2n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{2n-2} & x_{2n-3} & \dots & x_0 & x_1 \\ x_{2n-1} + x_1 & x_{2n-2} + x_2 & \dots & x_1 + x_{2n-1} & x_0 + x_{2n} \end{pmatrix}$$

and

$$A''_{2n} = \begin{pmatrix} x_0 & x_1 & \dots & x_{2n-2} & x_{2n-1} \\ x_1 & x_0 & \dots & x_{2n-3} & x_{2n-2} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{2n-2} & x_{2n-3} & \dots & x_0 & x_1 \\ x_{2n-1} - x_1 & x_{2n-2} - x_2 & \dots & x_1 - x_{2n-1} & x_0 - x_{2n} \end{pmatrix}.$$

Then

$$\det A'_{2n} + \det A''_{2n} = 2 \det A_{2n} = 2a_{2n}.$$

Define

$$J'_{2n} = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{n-1} & 0_{(n-1) \times 1} & -\frac{1}{\sqrt{2}} L_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 1 & 0_{1 \times (n-1)} & 0 \\ \frac{1}{\sqrt{2}} L_{n-1} & 0_{(n-1) \times 1} & \frac{1}{\sqrt{2}} I_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 & 0_{1 \times (n-1)} & 1 \end{pmatrix}$$

and

$$J''_{2n} = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{n-1} & 0_{(n-1) \times 1} & \frac{1}{\sqrt{2}} L_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 1 & 0_{1 \times (n-1)} & 0 \\ -\frac{1}{\sqrt{2}} L_{n-1} & 0_{(n-1) \times 1} & \frac{1}{\sqrt{2}} I_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0 & 0_{1 \times (n-1)} & 1 \end{pmatrix}.$$

By matrix calculation, we get

$$J'_{2n} A'_{2n} (J'_{2n})^{-1} = \begin{pmatrix} X_1 & *_1 \\ 0_{(n+1) \times (n-1)} & X_2 \end{pmatrix}$$

with $\det X_1 = c_{n-1}$ and $\det X_2 = d_{n+1}$. Similarly we get

$$J''_{2n} A''_{2n} (J''_{2n})^{-1} = \begin{pmatrix} Y_2 & *_2 \\ 0_{n \times n} & Y_1 \end{pmatrix}$$

with $\det Y_1 = c_n$ and $\det Y_2 = d_n$. Taking determinants, we get

$$2a_{2n} = c_n d_n + c_{n-1} d_{n+1}.$$

Let

$$A'_{2n+1} = \begin{pmatrix} x_0 & x_1 & \dots & x_{2n-2} & x_{2n} \\ x_1 & x_0 & \dots & x_{2n-2} & x_{2n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{2n-1} & x_{2n-2} & \dots & x_0 & x_1 \\ x_{2n} + x_1 & x_{2n-1} + x_2 & \dots & x_1 + x_{2n} & x_0 + x_{2n+1} \end{pmatrix}$$

and

$$A''_{2n+1} = \begin{pmatrix} x_0 & x_1 & \dots & x_{2n-1} & x_{2n} \\ x_1 & x_0 & \dots & x_{2n-2} & x_{2n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{2n-1} & x_{2n-2} & \dots & x_0 & x_1 \\ x_{2n} - x_1 & x_{2n-1} - x_2 & \dots & x_1 - x_{2n} & x_0 - x_{2n+1} \end{pmatrix}.$$

Then we have

$$\det A'_{2n+1} + \det A''_{2n+1} = 2 \det A_{2n+1} = 2a_{2n+1}.$$

Let

$$J'_{2n+1} = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{n-1} & -\frac{1}{\sqrt{2}} L_{n-1} & 0_{(n-1) \times 1} \\ \frac{1}{\sqrt{2}} L_{n-1} & \frac{1}{\sqrt{2}} I_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}$$

and

$$J''_{2n+1} = \begin{pmatrix} \frac{1}{\sqrt{2}} I_{n-1} & \frac{1}{\sqrt{2}} L_{n-1} & 0_{(n-1) \times 1} \\ -\frac{1}{\sqrt{2}} L_{n-1} & \frac{1}{\sqrt{2}} I_{n-1} & 0_{(n-1) \times 1} \\ 0_{1 \times (n-1)} & 0_{1 \times (n-1)} & 1 \end{pmatrix}.$$

By matrix calculation, we get

$$J'_{2n+1} A'_{2n+1} (J'_{2n+1})^{-1} = \begin{pmatrix} X_1 & *_1 \\ 0_{(n+1) \times (n-1)} & X_2 \end{pmatrix}$$

with $\det X_1 = b_n$ and $\det X_2 = b'_{n+1}$. Similarly we get

$$J''_{2n+1} A''_{2n+1} (J''_{2n+1})^{-1} = \begin{pmatrix} Y_2 & *_2 \\ 0_{n \times n} & Y_1 \end{pmatrix}$$

with $\det Y_1 = b_{n+1}$ and $\det Y_2 = b'_n$. Taking determinants, we get

$$2a_{2n+1} = b_n b'_{n+1} + b_{n+1} b'_n.$$

□

Remark 9.4. Here, we remark that, the equations (3)-(7) are discovered in [AYY] and the equalities

$$\det A_{2n} = \det B_n \det B'_n,$$

$$\det A_{2n+1} = \det C_n \det D'_{n+1},$$

$$2 \det A_{2n} = \det C_n \det D'_n + \det C_{n-1} \det D'_{n+1}$$

and

$$2 \det A_{2n+1} = \det B_n \det B'_{n+1} + \det B_{n+1} \det B'_n$$

are proven in the book [VD], Page 88.

Proposition 9.5. *The equalities $a_{2n} = b_n b'_n$, $2a_{2n} = c_n d_n + c_{n-1} d_{n+1}$, $a_{2n+1} = c_n d_{n+1}$, $2a_{2n+1} = b_n b'_{n+1} + b'_n b_{n+1}$ generate all algebraic relations among*

$$\{a_n, b_n, b'_n, c_n, d_n | n \geq 1\}.$$

Proof. By Proposition 8.8, we know that $\{b_n, c_n | n \geq 1\}$ are algebraically independent. It is clear that, from these equalities we can express a_n, b'_n, d_n in terms of rational functions of $\{b_m, c_m | 1 \leq m \leq n\}$, so these equalities generate all algebraic relations among $\{a_n, b_n, b'_n, c_n, d_n | n \geq 1\}$. \square

Remark 9.6. *More important than Proposition 9.5 itself is: we can use the four identities to derive many other algebraic relations. Moreover, by expressing a_n, b'_n, d_n in terms of rational functions of $\{b_m, c_m | 1 \leq m \leq n\}$, we are able to check whether any given polynomial function of $\{a_n, b_n, b'_n, c_n, d_n | n \geq 1\}$ is identical to 0 or not.*

Type BC. From the equations $b_{n+1} b'_n + b_n b_{n+1} = 2c_n d_{n+1}$, $2b_n b'_n = c_n d_n + c_{n-1} d_{n+1}$, $2b_{n+1} b'_{n+1} = c_{n+1} d_{n+1} + c_n d_{n+2}$, we get

$$(12) \quad b_{n+1}^2 (c_n d_n + c_{n-1} d_{n+1}) + b_n^2 (c_{n+1} d_{n+1} + c_n d_{n+2}) - 4b_{n+1} b_n c_n d_{n+1} = 0.$$

When $n = 0$, this equation is $2b_1^2 + (c_1 d_1 + d_2) - 4b_1 d_1 = 0$. Since $\{d_n, c_n | n \geq 1\}$ are algebraically independent, so these equations generate all algebraic relations among $\{b_n, c_n, d_n | n \geq 1\}$. Together with the identities $a_{2n-1} - c_{n-1} d_n = 0$ and $2a_{2n} - c_n d_n - c_{n-1} d_{n+1} = 0$, they generate all algebraic relations among $\{a_n, b_n, c_n, d_n | n \geq 1\}$.

Type B. From $2a_{2n+1} = b_{n+1} b'_n + b_n b_{n+1}$, $a_{2n+2} = b_{n+1} b'_{n+1}$, $a_{2n} = b_n b'_n$, we get

$$(13) \quad a_{2n+2} b_n^2 + a_{2n} b_{n+1}^2 = 2a_{2n+1} b_n b_{n+1}.$$

From $2a_{2n} = c_n d_n + c_{n-1} d_{n+1}$, $a_{2n+1} = c_n d_{n+1}$, $a_{2n-1} = c_{n-1} d_n$, we get

$$(14) \quad a_{2n+1} d_n^2 + a_{2n-1} d_{n+1}^2 = 2a_{2n} d_n d_{n+1}.$$

The equalities

$$\{a_{2n+2} b_n^2 + a_{2n} b_{n+1}^2 - 2a_{2n+1} b_n b_{n+1} = 0 | n \geq 0\}$$

and

$$\{a_{2n+1} d_n^2 + a_{2n-1} d_{n+1}^2 - 2a_{2n} d_n d_{n+1} = 0 | n \geq 1\}$$

generate all algebraic relations among $\{a_n, b_n, d_n | n \geq 1\}$. Here we regard $a_{-1} = d_1^{-1} = x_0^{-1}$, $a_0 = 1$.

Type C. We have

$$(15) \quad a_{2n+1} = c_n d_{n+1}$$

and

$$(16) \quad 2a_{2n+2} = c_{n+1} d_{n+1} + c_n d_{n+2}$$

As $\{d_n, c_n | n \geq 1\}$ are algebraically independent, the equalities

$$\{a_{2n+1} - c_n d_{n+1} = 0 | n \geq 1\}$$

and

$$\{2a_{2n+2} - c_{n+1} d_{n+1} - c_n d_{n+2} = 0 | n \geq 0\}$$

generate all algebraic relations among $\{a_n, c_n, d_n | n \geq 1\}$.

Type D. The equations

$$\{a_{2n+1} d_n^2 + a_{2n-1} d_{n+1}^2 - 2a_{2n} d_n d_{n+1} = 0 | n \geq 1\}$$

generate all algebraic relations among $\{a_n, d_n | n \geq 1\}$.

Remark 9.7. *Let*

$$R = \mathbb{Q}[\{y_{1,n}, y_{2,n}, y_{3,n}, y_{4,n}, y_{5,n} \mid n \geq 1\}]$$

be the free polynomial algebra over the rational field with infinitely many indeterminates

$$\{y_{1,n}, y_{2,n}, y_{3,n}, y_{4,n}, y_{5,n} \mid n \geq 1\}.$$

Let

$$\text{Eva} : R \longrightarrow \mathbb{Q}[\{x_n \mid n \geq 1\}]$$

be the homomorphism defined by

$$\text{Eva}(x_{1,n}) = a_n, \text{Eva}(x_{2,n}) = b_n, \text{Eva}(x_{3,n}) = b'_n$$

and

$$\text{Eva}(x_{4,n}) = c_n, \text{Eva}(x_{5,n}) = d_n.$$

Let I be the kernel of I . Let

$$(17) \quad f_{1,n} = y_{1,2n} - y_{2,n}y_{3,n},$$

$$(18) \quad f_{2,n} = y_{1,2n+1} - y_{4,n}y_{5,n+1},$$

$$(19) \quad f_{3,n} = 2y_{1,2n} - y_{4,n}y_{5,n} - y_{4,n-1}y_{5,n+1}$$

and

$$(20) \quad f_{4,n} = 2y_{1,2n+1} - y_{2,n}y_{3,n+1} - y_{3,n}y_{2,n+1}.$$

Then $f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n} \in I$ for any $n \geq 1$. Moreover, let R_1 be the localization of R with respect to the multiplicative system generated by $\{y_{2,n}, y_{4,n} \mid n \geq 1\}$. Then we have a homomorphism $\text{Eva}_1 : R_1 \longrightarrow \mathbb{Q}(\{x_n \mid n \geq 1\})$. One can show that, the elements $\{f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n} \mid n \geq 1\}$ generate the kernel of Eva_1 . Similarly, they generate the corresponding kernels if we consider the localization with respect to the multiplicative system generated by $\{y_{3,n}, y_{4,n} \mid n \geq 1\}$ (or $\{y_{2,n}, y_{4,n} \mid n \geq 1\}$, etc) and the similar homomorphism. Does $\{f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n} \mid n \geq 1\}$ generate I ? It seems to the author that the answer to this is no. In that case, can one find a system of generators of I ? One could consider a subset of $\{a_n, b_n, b'_n, c_n, d_n \mid n \geq 1\}$ (e.g. $\{a_n, b_n, c_n, d_n \mid n \geq 1\}$), the similar homomorphism as the above Eva and ask about the generators of the kernel.

9.2. Classical irreducible root systems.

Proposition 9.8. *Given a root system Ψ and an automorphism $\gamma \in \text{Aut}(\Psi)$, if $\gamma\Phi = \Phi$ for a reduced sub-root system Φ of Ψ , then $\gamma F_{\Phi, W_\Psi} = F_{\Phi, W_\Psi}$.*

Proof. Replacing γ by some $w\gamma$ ($w \in W_\Phi$) if necessary, we may assume that $\gamma\Phi^+ = \Phi^+$. Then $\gamma\delta_\Phi = \delta_\Phi$ and γ maps simple roots of Φ to simple roots. By the latter, we get that $\gamma W_\Phi \gamma^{-1} = W_\Phi$.

Hence

$$\begin{aligned} \gamma F_{\Phi, W_\Psi} &= \gamma \left(\sum_{w \in W_\Phi} \chi_{\delta_\Phi - w\delta_\Phi, W_\Psi}^* \right) \\ &= \sum_{w \in W_\Phi} \chi_{\gamma\delta_\Phi - (\gamma w \gamma^{-1})\gamma\delta_\Phi, W_\Psi}^* \\ &= \sum_{w \in W_\Phi} \chi_{\delta_\Phi - (\gamma w \gamma^{-1})\delta_\Phi, W_\Psi}^* \\ &= \sum_{w \in W_\Phi} \chi_{\delta_\Phi - w\delta_\Phi, W_\Psi}^* \\ &= F_{\Phi, W_\Psi}. \end{aligned}$$

□

Given a classical irreducible root system Ψ of rank n , in the case that $\Psi = A_n$ ($n \geq 1$), we have the following simple statement.

Proposition 9.9. *For any sub-root system $\Phi \subset A_n$, $F_{\Phi, W_{A_n}} = F_{\Phi, \text{Aut}(A_n)}$.*

For any two sub-root systems $\Phi_1, \Phi_2 \subset A_n$, $2\delta'_{\Phi_1} = 2\delta'_{\Phi_2}$ if and only if $\Phi_1 \sim \Phi_2$.

For any pairwise non-conjugate sub-root systems $\Phi_1, \dots, \Phi_s \subset A_n$, the characters $\{F_{\Phi_i, W_{A_n}} | 1 \leq i \leq s\}$ are linearly independent.

Proof. By Proposition 9.8, we have $F_{\Phi, W_{A_n}} = F_{\Phi, \text{Aut}(A_n)}$ since there exists $\gamma \in \text{Aut}(A_n) - W_{A_n}$ such that $\gamma\Phi = \Phi$ (the index of W_{A_n} in $\text{Aut}(A_n)$ is 2).

For a sub-root system

$$\Phi \cong \bigsqcup_{1 \leq i \leq s} A_{n_i-1} \subset A_n$$

with $n_i \geq 1$ and $\sum_{1 \leq i \leq s} n_i = n + 1$,

$$2\delta_\Phi \sim (\underbrace{n_1 - 1, n_1 - 3, \dots, 1 - n_1}_{}, \dots, \underbrace{n_s - 1, n_s - 3, \dots, 1 - n_s}_{}).$$

From this one sees that the weights $2\delta_\Phi$ are non-conjugate for any two non-conjugate sub-root systems. Therefore the remaining conclusions in the proposition follow. \square

In the case that $\Psi = BC_n, B_n$ or C_n , we have $W_\Psi = W_{BC_n} = W_n$. Therefore, the question of getting linear relations among $\{F_{\Phi, W_\Psi} | \Phi \subset \Psi\}$ reduces to the question of getting algebraic relations of homogeneous degree n among the polynomials $\{a_m, b_m, c_m, d_m | 1 \leq m \leq n\}$, $\{a_m, b_m, d_m | 1 \leq m \leq n\}$, $\{a_m, c_m, d_m | 1 \leq m \leq n\}$, respectively. These algebraic relations are discussed in the last subsection.

In the case of $\Psi = BC_n$, we make an interesting observation. We have showed the equations $b_{n+1}^2(c_n d_n + c_{n-1} d_{n+1}) + b_n^2(c_{n+1} d_{n+1} + c_n d_{n+2}) - 4b_{n+1} b_n c_n d_{n+1} = 0$ for any $n \geq 0$. When $n = 0$ and $n = 1$, they are $2b_1^2 + (c_1 d_1 + d_2) - 4b_1 d_1 = 0$ and

$$b_2^2(c_1 d_1 + d_2) + b_1^2(c_2 d_2 + c_1 d_3) - 4b_2 b_1 c_1 d_2 = 0.$$

Eliminating d_1 , we get

$$\begin{aligned} 0 &= (4b_1 - c_1)(b_2^2(c_1 d_1 + d_2) + b_1^2(c_2 d_2 + c_1 d_3) - 4b_2 b_1 c_1 d_2) \\ &\quad + (b_2^2 c_1)(2b_1^2 + (c_1 d_1 + d_2) - 4b_1 d_1) \\ &= b_1(2b_1 b_2^2 c_1 + 4b_1^2 c_2 d_2 + 4b_1^2 c_1 d_3 + 4b_2 c_1^2 d_2 + 4b_2^2 d_2 \\ &\quad - b_1 c_1 c_2 d_2 - b_1 c_1^2 d_3 - 16b_1 b_2 c_1 d_2). \end{aligned}$$

Since $b_1 = x_0 - x_1$ is irreducible, we get

$$2b_1 b_2^2 c_1 + 4b_1^2 c_2 d_2 + 4b_1^2 c_1 d_3 + 4b_2 c_1^2 d_2 + 4b_2^2 d_2 - b_1 c_1 c_2 d_2 - b_1 c_1^2 d_3 - 16b_1 b_2 c_1 d_2 = 0.$$

This gives 8 rank-6 semisimple subgroups of $SU(15)$ with linearly dependent dimension data. This equation is also given in [AYY], Example 5.6 and implicit in [LP], Page 393.

In the case of $\Psi = D_n$ ($n \geq 4$), algebraic relations of homogeneous degree n among $\{a_m, c_m, d_m | 1 \leq m \leq n\}$ correspond to the linear relations among the characters

$$\{F_{\Phi, W_n} | \Phi \subset D_n\}.$$

Note that

$$W_{D_n} = \Gamma_n \rtimes S_n$$

is a subgroup of W_n of index 2, where

$$\Gamma_n = \{(a_1, a_2, \dots, a_n) \in \{\pm 1\}^n : a_1 a_2 \cdots a_n = 1\}.$$

Proposition 9.10. *Given $\Psi = D_n$ ($n \geq 4$) and a sub-root system $\Phi \subset \Psi$, the following conditions are equivalent to each other:*

- (1) $F_{\Phi, W_\Psi} \neq F_{\Phi, W_n}$.
- (2) $2\delta'_\Phi$ is not W_n invariant.
- (3)

$$\Phi \cong \bigsqcup_{1 \leq i \leq s} A_{n_i-1}$$

with $2 \leq n_1 \leq n_2 \leq \dots \leq n_s \leq n$, each n_i is even, and $n_1 + n_2 + \dots + n_s = n$.

- (4) $\Phi \not\sim_{W_{D_n}} \gamma\Phi$ for some (equivalently, for any) $\gamma \in W_n - W_{D_n}$.

Proof. (2) \Rightarrow (1) is clear. We prove (1) \Rightarrow (3), (3) \Rightarrow (2) and (3) \Leftrightarrow (4) in the below.

Any sub-root system Φ of D_n (cf. [LP] and [Os]) is conjugate to one of

$$\left(\bigsqcup_{1 \leq i \leq s} D_{n_i} \right) \bigsqcup \left(\bigsqcup_{1 \leq j \leq t} A_{m_j} \right),$$

where $n_i \geq 1$, $m_j \geq 2$, $\sum_{1 \leq i \leq s} n_i + \sum_{1 \leq j \leq t} m_j = n$. Thus (3) \Leftrightarrow (4).

If (3) does not hold, then $s \geq 1$ or $s = 0$ and some m_i is odd. In the case that $s \geq 1$, we may assume that $D_{n_1} = \langle e_1 - e_2, \dots, e_{n_1-1} - e_{n_1}, e_{n_1-1} + e_{n_1} \rangle \subset \Phi$. Thus $s_1\Phi = \Phi$. By Proposition 9.8 we get $s_1F_{\Phi, W_\Psi} = F_{\Phi, W_\Psi}$. Since s_1, W_{D_n} generate $\text{Aut}(D_n)$ ($n \geq 5$), we have

$$F_{\Phi, W_\Psi} = F_{\Phi, W_n}.$$

In the case that $s = 0$ and some m_j is odd, we may and do assume that m_1 is odd and $A_{m_1} = \langle e_1 - e_2, \dots, e_{m_1-1} - e_{m_1} \rangle$. Then $s_1s_2 \dots s_{m_1}\Phi = \Phi$. By Proposition 9.8,

$$s_1s_1 \dots s_{m_1}F_{\Phi, W_\Psi} = F_{\Phi, W_\Psi}.$$

Since $s_1s_2 \dots s_{m_1}, W_{D_n}$ generate $\text{Aut}(D_n)$ ($n \geq 5$), we get $F_{\Phi, W_\Psi} = F_{\Phi, W_n}$. This proves (1) \Rightarrow (3).

If (3) holds, by calculation we get

$$2\delta_\Phi = (n_1 - 1, n_1 - 3, \dots, 1 - n_1, \dots, n_s - 1, \dots, 3 - n_s, 1 - n_s).$$

Thus $2\delta'_\Phi$ is not W_n invariant. This proves (3) \Rightarrow (2). \square

By Proposition 9.10, for any sub-root system Φ of D_n , either $W_{D_n}\Phi = W_n\Phi$ and $F_{\Phi, W_{D_n}} = F_{\Phi, W_n}$, or $W_{D_n}\Phi \neq W_n\Phi$ and $F_{\Phi, W_{D_n}}, F_{\gamma\Phi, W_{D_n}}$ are linearly independent. Here γ is any element in $W_n - W_{D_n}$. In this way we get all linear relations among $\{F_{\Phi, W_{D_n}} : \Phi \subset D_n\}$ from the linear relations among $\{F_{\Phi, W_n} : \Phi \subset D_n\}$.

In the case of $\Psi = D_4$, its automorphism group is larger than W_n . We discuss more on this case. Firstly the W_Ψ -conjugacy classes of sub-root systems are

$$\emptyset, A_1, A_2, D_2, 2A_1, (2A_1)', A_3$$

and

$$(A_3)', D_3, D_2 + A_1, D_4, 2D_2.$$

Here

$$\begin{aligned} D_2 &= \langle e_1 - e_2, e_1 + e_2 \rangle, \\ 2A_1 &= \langle e_1 - e_2, e_3 - e_4 \rangle, \\ (2A_1)' &= \langle e_1 - e_2, e_3 + e_4 \rangle, \\ A_3 &= \langle e_1 - e_2, e_2 - e_3, e_3 - e_4 \rangle, \\ (A_3)' &= \langle e_1 - e_2, e_2 - e_3, e_3 + e_4 \rangle \end{aligned}$$

and

$$D_3 = \langle e_2 - e_3, e_3 - e_4, e_3 + e_4 \rangle.$$

The sub-root systems $D_2, 2A_1, (2A_1)'$ are conjugate to each other under $\text{Aut}(D_4)$, and the sub-root systems $A_3, (A_3)', D_3$ are conjugate to each other under $\text{Aut}(D_4)$.

Only the characters of sub-root systems A_2 and $4A_1$ have equal leading term, which is $2\omega_2$. We have

$$\begin{aligned} F_{A_2, W_\Psi} &= 1 - 2\chi_{\omega_2}^* + 2\chi_{\omega_1+\omega_3+\omega_4}^* - \chi_{2\omega_2}^*, \\ F_{4A_1, W_\Psi} &= 1 - 4\chi_{\omega_2}^* + 2(\chi_{2\omega_1}^* + \chi_{2\omega_3}^* + \chi_{2\omega_4}^*) - 4\chi_{\omega_1+\omega_3+\omega_4}^* + \chi_{2\omega_2}^*. \end{aligned}$$

A little more calculation shows that

$$F_{A_2, W_\Psi} + F_{4A_1, W_\Psi} - 2F_{D_2+A_1, W_\Psi} = 0.$$

This equality also follows from the equality $a_3 + d_2^2 = d_2(c_1 + d_2) = 2a_2d_2$ by noting that $F_{\Phi, W_{D_4}} = F_{\Phi, W_4}$ if $\Phi = A_2, 4A_1$ or $D_2 + A_1$. This is the only linear relation among $\{F_{\Phi, W_{D_4}} \mid \Phi \subset D_4\}$.

Proposition 9.11. *Given a compact connected simple Lie group G of type B_n ($n \geq 4$), C_n ($n \geq 3$) or D_n ($n \geq 4$), there exist non-isomorphic closed connected subgroups of G with linearly dependent dimension data.*

Proof. Taking $n = 1$ in Equation (14) and using $a_1 = d_1 = x_0$, we get

$$a_3d_1 + d_2^2 - 2a_2d_2 = 0.$$

This gives us a linear relation of three non-isomorphic subgroups of G in the case that G is of type B_n or D_n with $n \geq 4$. In the D_n case, by Proposition 9.10 we know that the W_{D_n} trace and the W_n trace of characters are equal for the sub-root systems corresponding to the polynomials a_3d_1 , d_2^2 and a_2d_2 .

By Equations (15) and (16), we get

$$2a_{2n+2}c_nc_{n+1} - c_{n+1}^2a_{2n+1} - c_n^2a_{2n+3} = 0.$$

Taking $n = 0$, we get $2a_2c_1 - a_1c_1^2 - a_3 = 0$, which gives us a linear relation of three non-isomorphic subgroups of G in the case that G is of type C_n with $n \geq 3$. \square

9.3. A generating function. Given an irreducible root system Ψ with a positive system Ψ^+ , a root $\alpha \in \Psi$ is called a short root if for any other root $\beta \in \Psi$, either $(\alpha, \beta) = 0$ or $|\beta| \geq |\alpha|$. We normalize the inner product on Ψ (or to say, on Λ_Ψ) by letting all the short roots of Ψ have length 1. For a reduced sub-root system Φ of Ψ , recall that

$$\delta_\Phi = \frac{1}{2} \sum_{\alpha \in \Phi \cap \Psi^+} \alpha$$

and δ'_Φ be the unique dominant weight in the Weyl group (of Ψ) orbit of δ_Φ . Let $e(\Phi) = |2\delta'_\Phi|^2$ be the square of the length of $2\delta'_\Phi$.

Definition 9.12. *For a reduced sub-root system Φ of Ψ , let*

$$f_{\Phi, \Psi}(t) = \sum_{w \in W_\Phi} \epsilon(w) t^{|\delta_\Phi - w\delta_\Phi|^2}.$$

In particular let $f_\Psi(t) = f_{\Psi, \Psi}(t)$.

As $|\delta_\Phi - w\delta_\Phi|^2 = (2\delta_\Phi, \delta_\Phi - w\delta_\Phi)$, an equivalent definition for f_Φ is

$$f_\Phi(t) = \sum_{w \in W_\Phi} \epsilon(w) t^{(2\delta_\Phi, \delta_\Phi - w\delta_\Phi)}.$$

If $\Phi = \bigsqcup_{1 \leq i \leq s} \Phi_i$ is an orthogonal decomposition of Φ into irreducible sub-root systems and $\sqrt{r_i}$ is the shortest length of roots in $\Phi_i \subset \Psi$, then

$$f_{\Phi, \Psi} = \prod_{1 \leq i \leq s} f_{\Phi_i}(t^{r_i}).$$

Thus the calculation of the polynomials $f_{\Phi, \Psi}(t)$ reduces to the calculation of the polynomials $f_{\Psi}(t)$ for irreducible root systems.

Definition 9.13. Let $\Lambda_{\Psi} \subset \mathbb{Q}\Psi$ be the set of integral weights of Ψ . For any weight $\lambda \in \Lambda_{\Psi}$, define

$$\chi_{\lambda}^* = \frac{1}{|W_{\Psi}|} \sum_{\gamma \in W_{\Psi}} \gamma \lambda.$$

We define a linear map

$$E' = \mathbb{Q}[\Lambda_{\Psi}]^{W_{\Psi}} \longrightarrow \mathbb{Q}[t]$$

by $E'(\chi_{\lambda}^*) = t^{|\lambda|^2}$.

Proposition 9.14. Given a reduced sub-root system Φ of Ψ , we have $E'(F_{\Phi, W_{\Psi}}) = f_{\Phi, \Psi}$.

Proof. This follows from the formulas

$$F_{\Phi, W_{\Psi}} = \sum_{w \in W_{\Phi}} \epsilon(w) \chi_{\delta_{\Phi} - w\delta_{\Phi}}^*$$

and

$$f_{\Phi, \Psi}(t) = \sum_{w \in W_{\Phi}} \epsilon(w) t^{|\delta_{\Phi} - w\delta_{\Phi}|^2}.$$

□

Let $\psi : \mathbb{Q}[x_0, x_1, \dots, x_n, \dots] \longrightarrow \mathbb{Q}[t]$ be an algebra homomorphism defined by

$$\psi(x_n) = t^{n^2}, \forall n \geq 0.$$

Proposition 9.15. Given $\Psi = \text{BC}_n$ and a reduced sub-root system Φ of BC_n , we have

$$f_{\Phi, \Psi} = \psi(E(j_n(F_{\Phi, W_n}))).$$

Proof. This follows from the definitions of E , ψ and $f_{\Phi, \Psi}$. □

Recall that $E(j_n(F_{\Phi, W_n})) \in \mathbb{Q}[x_0, x_1, \dots, x_n, \dots]$ is the multi-variable polynomial associated to reduced sub-root systems of BC_n in Section 8. Proposition 9.15 connects two polynomials by a simple relation.

For irreducible reduced root systems of small rank, calculation shows that $f_{A_1} = 1 - t$,

$$f_{A_2} = (1 - t)^2(1 - t^2),$$

$$f_{B_2} = (1 - t)(1 - t^2)(1 - t^3)(1 - t^4),$$

$$f_{G_2} = (1 - t)(1 - t^3)(1 - t^4)(1 - t^5)(1 - t^6)(1 - t^9),$$

and $f_{A_3} = (1 - t)^3(1 - t^2)^2(1 - t^3)$. By calculation, we also have

$$f_{A_4} = 1 - 4t + 3x^2 + 6x^3 - 7x^4 - 2x^5 - 4x^6 + \text{higher terms},$$

$$f_{D_4} = 1 - 4t + 3x^2 + 5x^3 - 3x^4 - 6x^5 - 6x^6 + \text{higher terms},$$

$$f_{A_5} = 1 - 5t + 6x^2 + 7x^3 - 16x^4 + 0x^5 + 2x^6 + \text{higher terms}.$$

Proposition 9.16. $f_{\Phi, \Psi}$ has constant term 1 and leading term $(-1)^{|\Phi^+|} t^{2\delta_{\Phi}^2}$, it satisfies

$$f_{\Phi, \Psi}(t) = (-1)^{|\Phi^+|} t^{2\delta_{\Phi}^2} f_{\Phi, \Psi}(t^{-1}).$$

Proof. Recall that, W_{Φ} has a longest element ω_0 which maps Φ^+ to $-\Phi^+$, so $\epsilon_{\omega_0} = (-1)^{|\Phi^+|}$ and $w_0(\delta_{\Phi}) = -\delta_{\Phi}$. Moreover, For any $w \in W$, we have

$$(2\delta_{\Phi}, \delta_{\Phi} - w\delta_{\Phi}) + (2\delta_{\Phi}, \delta_{\Phi} - w_0 w^{-1} \delta_{\Phi}) = |2\delta_{\Phi}|^2.$$

Therefore the proposition follows. □

Proposition 9.17. *Given an irreducible reduced root system Ψ , we have*

$$f_\Psi = \prod_{\alpha \in \Psi^+} (1 - t^{(2\delta_\Psi, \alpha)}).$$

Proof. Let $E'' : \mathbb{Q}[\Lambda_\Psi] \rightarrow \mathbb{Q}[t]$ be a linear map defined by

$$E''([\lambda]) = t^{\langle 2\delta_\Psi, \lambda \rangle}, \quad \forall \lambda \in \Lambda_\Psi.$$

Then E'' is an algebra homomorphism and we have

$$f_\Psi(t) = E''\left(\sum_{w \in W_\Psi} \text{sign}(w)[\delta_\Psi - w\delta_\Psi]\right).$$

By the Weyl denominator formula, we have

$$\sum_{w \in W_\Psi} \text{sign}(w)[w\delta_\Psi] = \prod_{\alpha \in \Psi^+} \left(\left[\frac{\alpha}{2}\right] - \left[-\frac{\alpha}{2}\right]\right).$$

Then we have

$$\sum_{w \in W_\Psi} \text{sign}(w)[\delta_\Psi - w\delta_\Psi] = \prod_{\alpha \in \Psi^+} (1 - [\alpha]).$$

Taking the map E'' on both sides, we get

$$f_\Psi = \prod_{\alpha \in \Psi^+} (1 - t^{(2\delta_\Psi, \alpha)}).$$

□

9.4. Exceptional irreducible root systems. In this subsection, for any exceptional irreducible root system Ψ , we consider the linear relations among the characters $\{F_{\Phi, W_\Psi} \mid \Phi \subset \Psi\}$.

We start with some observations.

Firstly, if

$$\sum_{1 \leq i \leq s} c_i F_{\Phi_i, W_\Psi} = 0$$

for some reduced sub-root systems $\{\Phi_i \subset \Psi : 1 \leq i \leq s\}$ and some non-zero coefficients $\{c_i \in \mathbb{R} : 1 \leq i \leq s\}$, then for any i with

$$|\delta'_{\Phi_i}| = \max\{|\delta'_{\Phi_j}| : 1 \leq j \leq i\},$$

there exists $j \neq i$ such that $\delta'_{\Phi_j} = \delta'_{\Phi_i}$. Since otherwise, $\chi_{2\delta_{\Phi_i, W_\Psi}}^*$ has a non-zero coefficient in the character $\sum_{1 \leq i \leq s} c_i F_{\Phi_i, W_\Psi}$.

Secondly, if Φ_1, \dots, Φ_s are all contained in another reduced sub-root system Ψ' of Ψ , and $\sum_{1 \leq i \leq s} c_i F_{\Phi_i, W_{\Psi'}} = 0$ for some constants $c_1, \dots, c_s \in \mathbb{R}$, then $\sum_{1 \leq i \leq s} c_i F_{\Phi_i, W_\Psi} = 0$. This is due to

$$F_{\Phi_i, W_\Psi} = \frac{1}{|W_\Psi|} \sum_{\gamma \in W_\Psi} \gamma F_{\Phi_i, W_{\Psi'}}$$

for each i , $1 \leq i \leq s$.

Thirdly, if Φ'_1, \dots, Φ'_s are all contained in another reduced sub-root system Ψ' of Ψ , and

$$\sum_{1 \leq i \leq s} c_i F_{\Phi'_i, W_{\Psi'}} = 0$$

for some constants c_1, \dots, c_s . Let

$$\Phi' \subset \Psi'^\perp = \{\alpha \in \Psi \mid (\alpha, \beta) = 0, \forall \beta \in \Psi'\}$$

and $\Phi_i = \Phi'_i \sqcup \Phi'$. Then

$$\sum_{1 \leq i \leq s} c_i F_{\Phi_i, W_\Psi} = 0.$$

This follows from the second observation.

Fourthly, if $\sum_{1 \leq i \leq s} c_i F_{\Phi_i, W_\Psi} = 0$, then

$$\sum_{1 \leq i \leq s} c_i f_{\Phi_i, \Psi} = 0.$$

This follows by applying the map E' .

Type E_6 . In the proof of Theorem 8.14, we have observed that only the two weights $2\omega_2, 2\omega_4$ are of the form $2\delta'_\Phi$ for at least two non-conjugate reduced sub-root systems of E_6 . It happens that $2\delta'_\Phi = 2\omega_2$ for $\Phi = A_2, 4A_1$, and $2\delta'_\Phi = 2\omega_4$ for $\Phi = 3A_2, A_3 + 2A_1$.

Since $D_4 \subset E_6$, from the conclusion in D_4 case and the second observation, we get

$$(21) \quad F_{A_2, W_{E_6}} + F_{4A_1, W_{E_6}} - 2F_{3A_2, W_{E_6}} = 0.$$

Moreover, we have

$$(22) \quad F_{3A_2, W_{E_6}} + F_{A_3+2A_1, W_{E_6}} + F_{A_3+A_1, W_{E_6}} - 3F_{2A_2+A_1, W_{E_6}} = 0.$$

The proof of this equality is given below. These two relations generate all linear relations among $\{F_{\Phi, W_{E_6}} | \Phi \subset E_6\}$.

Let θ be a linear map on weights defined by $\theta(\omega_1) = \omega_6$, $\theta(\omega_6) = \omega_1$, $\theta(\omega_3) = \omega_5$, $\theta(\omega_5) = \omega_3$, $\theta(\omega_2) = \omega_2$, $\theta(\omega_4) = \omega_4$. Then θ acts as an isometry and it maps dominant integral weights to dominant integral weights. We have $\text{Aut}(E_6) = W_{E_6} \rtimes \langle \theta \rangle$ as groups acting on the weights.

Lemma 9.18. *For a positive integer $k \leq 11$, if $k \neq 4, 5, 7, 8, 9, 10$, then there exists a unique $\text{Aut}(E_6)$ -orbit of weights λ in the root lattice such that $|\lambda|^2 = k$.*

When $k = 4, 5, 7, 8, 9$, there exist two $\text{Aut}(E_6)$ -orbits of weights in the root lattice with $|\lambda|^2 = k$. The representatives are

$$k = 4 : \{\omega_1 + \omega_3, 2\omega_2\};$$

$$k = 5 : \{\omega_3 + \omega_5, \omega_1 + \omega_2 + \omega_6\};$$

$$k = 7 : \{2\omega_1 + \omega_5, \omega_2 + \omega_4\};$$

$$k = 8 : \{2\omega_1 + 2\omega_6, \omega_1 + \omega_2 + \omega_3\};$$

$$k = 9 : \{3\omega_2, \omega_1 + \omega_4 + \omega_6\}.$$

When $k = 10$, there exist four $\text{Aut}(E_6)$ -orbits of weights in the root lattice with $|\lambda|^2 = 10$. The representatives are

$$\omega_1 + 2\omega_5, 3\omega_1 + \omega_2, \omega_2 + \omega_3 + \omega_5, \omega_1 + 2\omega_2 + \omega_6.$$

Proof. The inverse to the Cartan matrix of E_6 is

$$\frac{1}{3} \times \begin{pmatrix} 4 & 3 & 5 & 6 & 4 & 2 \\ 3 & 6 & 6 & 9 & 6 & 3 \\ 5 & 6 & 10 & 12 & 8 & 4 \\ 6 & 9 & 12 & 18 & 12 & 6 \\ 4 & 6 & 8 & 12 & 10 & 5 \\ 2 & 3 & 4 & 6 & 5 & 4 \end{pmatrix}.$$

Given a dominant integral weight $\lambda = \sum_{1 \leq i \leq 6} a_i \omega_i$, $a_i \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{aligned} |\lambda|^2 &= \frac{1}{3}(2a_1^2 + 2a_6^2 + 5a_3^2 + 5a_5^2) + \\ &\quad \frac{1}{3}(2a_1a_6 + 8a_3a_5 + 5a_1a_3 + 5a_5a_6 + 4a_1a_5 + 4a_3a_6) + \\ &\quad (a_2^2 + 3a_4^2 + 3a_2a_4) + (2a_3 + 2a_5 + a_1 + a_6)(a_2 + 2a_4). \end{aligned}$$

We also have: λ is in the root lattice if and only if $3|a_1 + a_5 - a_3 - a_6$.

If $|\lambda|^2 \leq 11$, let $\lambda_1 = a_1\omega_1 + a_3\omega_3 + a_5\omega_5 + a_6\omega_6$ and $\lambda_2 = a_2\omega_2 + a_4\omega_4$. Then, $|\lambda_1|^2 \leq 11$ and $|\lambda_2|^2 \leq 11$.

Consider the weight λ_1 . Let $k = a_1 + a_5$ and $l = a_3 + a_6$, λ is in the root lattice implies that $3|k - l$. From $|\lambda_1|^2 \leq 11$, we get $k, l \leq 3$. When $|k - l| = 3$, we have

$$\lambda_1 = 3\omega_1(6), 3\omega_6(6), 2\omega_1 + \omega_5(7),$$

$$\omega_3 + 2\omega_6(7), \omega_1 + 2\omega_5(10), 2\omega_3 + \omega_6(10).$$

Here, the numbers in the brackets mean the squares of modulus of the weights. Similar notation will be used in the remaining part of this proof and the proof for Lemma 9.19. When $|k - l| = 0$, we have $k = l \leq 2$. Moreover, we have

$$\lambda_1 = 2\omega_1 + \omega_3 + \omega_6(11), \omega_1 + \omega_5 + 2\omega_6(11), 2\omega_1 + 2\omega_6(8),$$

$$\omega_3 + \omega_5(5), \omega_1 + \omega_3(4), \omega_5 + \omega_6(4), \omega_1 + \omega_6(2).$$

Consider the weight λ_2 . From $|\lambda_2|^2 \leq 11$, we get

$$\lambda_2 = \omega_4 + \omega_2(7), \omega_4(3), 3\omega_2(9), 2\omega_2(4), \omega_2(1).$$

In the case that $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, we have

$$\lambda = \omega_1 + 2\omega_2 + \omega_6(10), 3\omega_1 + \omega_2(10), \omega_2 + 3\omega_6(10),$$

$$\omega_2 + \omega_3 + \omega_5(10), \omega_1 + \omega_4 + \omega_6(9), \omega_1 + \omega_2 + \omega_3(8),$$

$$\omega_2 + \omega_5 + \omega_6(8), \omega_1 + \omega_2 + \omega_6(5).$$

We finish the proof of the lemma. \square

Proof of equality (22). Since any weight appearing in $F_{\Phi, W_{E_6}}$ is an integral linear combination of roots, so any term χ_λ^* appearing in $F_{\Phi, W_{E_6}}$ having λ in the root lattice. A case by case calculation enables us to show that any reduced sub-root system $\Phi \subset E_6$ is stable under the action of some $\gamma \in \text{Aut}(E_6) - W_{E_6}$, so $\theta F_{\Phi, W_{E_6}} = F_{\Phi, W_{E_6}}$ (cf. Proposition 9.8). Thus the coefficient of any term χ_λ^* in $F_{\Phi, W_{E_6}}$ is equal to the coefficient of the term $\chi_{\theta\lambda}^*$ in $F_{\Phi, W_{E_6}}$. By Lemma 9.18, to prove equality (22) it is enough to prove the corresponding equality about f_{Φ, E_6} and to calculate the coefficients of terms χ_λ^* with $|\lambda|^2 = 4, 5, 7, 8, 9, 10$.

For the functions f_{Φ, E_6} , we have

$$\begin{aligned} & f_{3A_2, E_6} + f_{A_3+2A_1, E_6} + f_{A_3+A_1, E_6} - 3f_{2A_2+A_1, E_6} \\ &= (1-t)^6(1-t^2)^3 + (1-t)^5(1-t^2)^2(1-t^3) \\ &\quad + (1-t)^4(1-t^2)^2(1-t^3) - 3(1-t)^5(1-t^2)^2 \\ &= (1-t)^4(1-t^2)^2((1-2t+2t^3-t^4) + (1-t-t^3+t^4) + (1-t^3)) \\ &\quad - 3(1-t)^5(1-t^2)^2 \\ &= (1-t)^4(1-t^2)^2(3-3t) - 3(1-t)^5(1-t^2)^2 \\ &= 0. \end{aligned}$$

The terms χ_λ^* with $|\lambda|^2 = 4, 5, 7, 8, 9, 10$ in

$$F_{3A_2, W_{E_6}} + F_{A_3+2A_1, W_{E_6}} + F_{A_3+A_1, W_{E_6}} - 3F_{2A_2+A_1, W_{E_6}}$$

are

$$\begin{aligned}
& (-12\chi_{\omega_1+\omega_3}^* - 12\chi_{\omega_5+\omega_6}^* - 3\chi_{2\omega_2}^*) + (-4\chi_{\omega_1+\omega_3}^* - 4\chi_{\omega_5+\omega_6}^* - \chi_{2\omega_2}^*) + \\
& (-2\chi_{\omega_1+\omega_3}^* - 2\chi_{\omega_5+\omega_6}^* - 2\chi_{2\omega_2}^*) + (-6\chi_{\omega_1+\omega_3}^* - 6\chi_{\omega_5+\omega_6}^* - 2\chi_{2\omega_2}^*) = 0, \\
& (36\chi_{\omega_1+\omega_2+\omega_6}^*) + (6\chi_{\omega_1+\omega_2+\omega_6}^*) + (0\chi_{\omega_1+\omega_2+\omega_6}^*) - 3(14\chi_{\omega_1+\omega_2+\omega_6}^*) = 0, \\
& (-12\chi_{2\omega_1+\omega_5}^* - 12\chi_{\omega_3+2\omega_6}^* - 12\chi_{\omega_2+\omega_4}^*) + (2\chi_{2\omega_1+\omega_5}^* + 2\chi_{\omega_3+2\omega_6}^* + 2\chi_{\omega_2+\omega_4}^*) + \\
& (\chi_{2\omega_1+\omega_5}^* + \chi_{\omega_3+2\omega_6}^* + 4\chi_{\omega_2+\omega_4}^*) - 3(-3\chi_{2\omega_1+\omega_5}^* - 3\chi_{\omega_3+2\omega_6}^* - 2\chi_{\omega_2+\omega_4}^*) = 0, \\
& (3\chi_{2\omega_1+2\omega_6}^* + 12\chi_{\omega_1+\omega_2+\omega_3}^* + 12\chi_{\omega_2+\omega_5+\omega_6}^*) + (-\chi_{2\omega_1+2\omega_6}^* - 4\chi_{\omega_1+\omega_2+\omega_3}^* - \\
& 4\chi_{\omega_2+\omega_5+\omega_6}^*) + (\chi_{2\omega_1+2\omega_6}^* - 2\chi_{\omega_1+\omega_2+\omega_3}^* - 2\chi_{\omega_2+\omega_5+\omega_6}^*) - \\
& 3(\chi_{2\omega_1+2\omega_6}^* + 2\chi_{\omega_1+\omega_2+\omega_3}^* + 2\chi_{\omega_2+\omega_5+\omega_6}^*) = 0, \\
& (2\chi_{3\omega_2}^*) + (-\chi_{3\omega_2}^*) + (-3\chi_{\omega_1+\omega_4+\omega_6}^* - \chi_{3\omega_2}^*) - 3(-\chi_{\omega_1+\omega_4+\omega_6}^*) = 0
\end{aligned}$$

and

$$\begin{aligned}
& (-3\chi_{\omega_1+2\omega_5}^* - 3\chi_{2\omega_3+\omega_6}^* - 6\chi_{\omega_1+2\omega_2+\omega_6}^*) + (2\chi_{\omega_1+2\omega_5}^* + 2\chi_{2\omega_3+\omega_6}^* + 4\chi_{\omega_1+2\omega_2+\omega_6}^*) \\
& + (\chi_{\omega_1+2\omega_5}^* + \chi_{2\omega_3+\omega_6}^* + 2\chi_{\omega_1+2\omega_2+\omega_6}^*) = 0
\end{aligned}$$

respectively.

Therefore the equality (22) follows. \square

Type E₇. As in the proof of Theorem 8.14, we have observed that those weights that appearing more than once in $\{2\delta'_\Phi \mid \Phi \subset E_7\}$ and the reduced sub-root systems for which they appeared in are as follows,

- (1) $2\omega_1$: $A_2, 4A_1'$, appears two times.
- (2) $\omega_1 + \omega_6$: $A_2 + A_1, 5A_1$, appears two times.
- (3) ω_4 : $A_2 + 2A_1, 6A_1$, appears two times.
- (4) $2\omega_2$: $A_2 + 3A_1, 7A_1$, appears two times.
- (5) $2\omega_3$: $3A_2, A_3 + 2A_1$, appears two times.
- (6) $2\omega_1 + 2\omega_6$: $A_4, 2A_3$, appears two times.
- (7) $\omega_1 + \omega_4 + \omega_6$: $A_4 + A_1, 2A_3 + A_1$, appears two times.

By the conclusion from the E₆ case and the second and the third observations in the beginning of this subsection, we get

$$(23) \quad F_{A_2, W_{E_7}} + F_{(4A_1)', W_{E_7}} - 2F_{3A_1, W_{E_7}} = 0,$$

$$(24) \quad F_{A_2+A_1, W_{E_7}} + F_{5A_1, W_{E_7}} - 2F_{4A_1, W_{E_7}} = 0,$$

$$(25) \quad F_{A_2+2A_1, W_{E_7}} + F_{6A_1, W_{E_7}} - 2F_{5A_1, W_{E_7}} = 0,$$

$$(26) \quad F_{A_2+3A_1, W_{E_7}} + F_{7A_1, W_{E_7}} - 2F_{6A_1, W_{E_7}} = 0,$$

$$(27) \quad F_{3A_2, W_{E_7}} + F_{A_3+2A_1, W_{E_7}} + F_{A_3+A_1, W_{E_7}} - 3F_{2A_2+A_1, W_{E_7}} = 0.$$

We prove that these relations generate all linear relations among $\{F_{\Phi, W_{E_7}} \mid \Phi \subset E_7\}$. By the first observation, to show this, we just need to show: any non-trivial linear combination

$$c_1 F_{A_4, W_{E_7}} + c_2 F_{2A_3, W_{E_7}}$$

is not a linear combination of the characters $\{F_{\Phi, W_{E_7}} : |\delta'_\Phi|^2 < |\delta'_{A_4}|^2 = 20\}$ and any non-trivial linear combination

$$c_1 F_{A_4+A_1, W_{E_7}} + c_2 F_{2A_3+A_1, W_{E_7}}$$

is not a linear combination of the characters $\{F_{\Phi, W_{E_7}} \mid |\delta'_\Phi|^2 < |\delta'_{A_4+A_1}|^2 = 21\}$.

Suppose

$$c_1 F_{A_4, W_{E_7}} + c_2 F_{2A_3, W_{E_7}} = \sum_{1 \leq i \leq s, |2\delta'_{\Phi_i}| \leq 19} c_i F_{\Phi_i, W_{E_7}}.$$

Then

$$c_1 f_{A_4, E_7} + c_2 f_{2A_3, E_7} = \sum_{1 \leq i \leq s, |2\delta'_{\Phi_i}| \leq 19} c_i f_{\Phi_i, E_7}$$

by the fourth observation. Since there is no $\Phi \subset E_7$ with $|2\delta'_{\Phi}|^2 = 19$, the RHS of the above equation has a degree at most 18. Comparing the coefficients of the terms t^{20} and t^{19} , we get $c_1 + c_2 = 0$ and $-4c_1 - 6c_2 = 0$. Hence $c_1 = c_2 = 0$.

After calculating the highest (=longest) terms, we get

$$\begin{aligned} F_{A_4+A_1, W_{E_7}} &= -\chi_{\omega_1+\omega_4+\omega_6}^* + 2\chi_{\omega_2+\omega_5+\omega_6}^* + \chi_{2\omega_1+2\omega_6}^* \\ &\quad + 2\chi_{\omega_1+\omega_2+\omega_3+\omega_7}^* + \text{lower terms}, \\ F_{2A_3+A_1, W_{E_7}} &= -\chi_{\omega_1+\omega_4+\omega_6}^* + 2\chi_{\omega_2+\omega_5+\omega_6}^* + \chi_{2\omega_1+2\omega_6}^* \\ &\quad + 4\chi_{\omega_1+\omega_2+\omega_3+\omega_7}^* + \text{lower terms}. \end{aligned}$$

Considering the coefficients of the terms $\chi_{\omega_1+\omega_4+\omega_6}^*$ and $\chi_{\omega_1+\omega_2+\omega_3+\omega_7}^*$, we see any non-trivial linear combination $c_1 F_{A_4+A_1, W_{E_7}} + c_2 F_{2A_3+A_1, W_{E_7}}$ is not a linear combination of the characters $\{F_{\Phi, W_{E_7}} : |\delta'_{\Phi}|^2 < |\delta'_{A_4+A_1}|^2 = 21\}$.

Type E_8 . As in the proof of Theorem 8.14, we have observed that those weights that appearing more than once in $\{2\delta'_{\Phi} \mid \Phi \subset E_8\}$ and the sub-root systems for which they appeared in are as follows,

- (1) $2\omega_8$: $A_2, 4A'_1$, appears two times.
- (2) $\omega_1 + \omega_8$: $A_2 + A_1, 5A_1$, appears two times.
- (3) ω_6 : $A_2 + 2A_1, 6A_1$, appears two times.
- (4) ω_3 : $A_2 + 3A_1, 7A_1$, appears two times.
- (5) $2\omega_1$: $2A_2, A_2 + 4A_1, 8A_1$, appears three times.
- (6) $2\omega_7$: $A_3 + 2A_1, 3A_2$, appears two times.
- (7) $\omega_2 + \omega_7$: $A_3 + 3A_1, 3A_2 + A_1$, appears two times.
- (8) $\omega_1 + \omega_6$: $A_3 + 4A_1, A_3 + A_2$, appears two times.
- (9) $2\omega_2$: $A_3 + A_2 + 2A_1, 4A_2$, appears two times.
- (10) $2\omega_1 + 2\omega_8$: $A_4, 2A_3$, appears two times.
- (11) $\omega_1 + \omega_6 + \omega_8$: $A_4 + A_1, 2A_3 + A_1$, appears two times.
- (12) $\omega_4 + \omega_8$: $A_4 + 2A_1, 2A_3 + 2A_1$, appears two times.
- (13) $2\omega_2 + 2\omega_8$: $D_4 + A_2, D_4 + 4A_1$, appears two times.
- (14) $2\omega_5$: $A_5 + A_2 + A_1, 2A_4$, appears two times.
- (15) $2\omega_1 + 2\omega_6$: $A_6, 2D_4$, appears two times.

By the conclusion from the E_7 case and the second and the third observations, we get

$$(28) \quad F_{A_2, W_{E_8}} + F_{(4A_1)', W_{E_8}} - 2F_{3A_1, W_{E_8}} = 0,$$

$$(29) \quad F_{A_2+A_1, W_{E_8}} + F_{5A_1, W_{E_8}} - 2F_{4A_1, W_{E_8}} = 0,$$

$$(30) \quad F_{A_2+2A_1, W_{E_8}} + F_{6A_1, W_{E_8}} - 2F_{5A_1, W_{E_8}} = 0,$$

$$(31) \quad F_{A_2+3A_1, W_{E_8}} + F_{7A_1, W_{E_8}} - 2F_{6A_1, W_{E_8}} = 0,$$

$$(32) \quad F_{A_2+4A_1, W_{E_8}} + F_{8A_1, W_{E_8}} - 2F_{7A_1, W_{E_8}} = 0,$$

$$(33) \quad F_{2A_2, W_{E_8}} + F_{A_2+4A_1, W_{E_8}} - 2F_{A_2+3A_1, W_{E_8}} = 0,$$

$$(34) \quad F_{3A_2, W_{E_8}} + F_{A_3+2A_1, W_{E_8}} + F_{A_3+A_1, W_{E_8}} - 3F_{2A_2+A_1, W_{E_8}} = 0,$$

$$(35) \quad F_{3A_2+A_1, W_{E_8}} + F_{A_3+3A_1, W_{E_8}} + F_{A_3+2A_1, W_{E_8}} - 3F_{2A_2+2A_1, W_{E_8}} = 0,$$

$$(36) \quad F_{4A_2, W_{E_8}} + F_{A_3+A_2+2A_1, W_{E_8}} + F_{A_3+A_2+A_1, W_{E_8}} - 3F_{3A_2+A_1, W_{E_8}} = 0,$$

$$(37) \quad F_{D_4+A_2, W_{E_8}} + F_{D_4+4A_1, W_{E_8}} - 2F_{D_4+3A_1, W_{E_8}} = 0.$$

In the below we prove that these relations generate all linear relations among $\{F_{\Phi, W_{E_8}} \mid \Phi \subset E_8\}$.

Since $\Psi = E_8$ has no sub-root systems Φ with $e(\Phi) = 19$, one can show similarly as in the E_7 case that any non-trivial linear combination $c_1 F_{A_4, W_{E_8}} + c_2 F_{2A_3, W_{E_8}}$ is not a linear combination of the characters $\{F_{\Phi, W_{E_8}} : |\delta'_{\Phi}|^2 < |\delta'_{A_4+A_1}|^2 = 20\}$.

By calculating the highest (=longest) terms, we get

$$\begin{aligned} F_{A_4+A_1, W_{E_8}} &= -\chi_{\omega_1+\omega_6+\omega_8}^* + 2\chi_{\omega_1+\omega_5}^* + \chi_{2\omega_1+2\omega_8}^* \\ &\quad + 2\chi_{\omega_2+\omega_7+\omega_8}^* + \text{lower terms}, \\ F_{2A_3+A_1, W_{E_8}} &= -\chi_{\omega_1+\omega_6+\omega_8}^* + 2\chi_{\omega_1+\omega_5}^* + \chi_{2\omega_1+2\omega_8}^* \\ &\quad + 4\chi_{\omega_2+\omega_7+\omega_8}^* + \text{lower terms}, \end{aligned}$$

Considering the coefficients of the terms $\chi_{\omega_1+\omega_6+\omega_8}^*$ and $\chi_{\omega_2+\omega_7+\omega_8}^*$, we see that any non-trivial linear combination $c_1 F_{A_4+A_1, W_{E_8}} + c_2 F_{2A_3+A_1, W_{E_8}}$ is not a linear combination of the characters $\{F_{\Phi, W_{E_8}} : |\delta'_{\Phi}|^2 < |\delta'_{A_4+A_1}|^2 = 21\}$.

We have

$$\begin{aligned} F_{A_4+2A_1, W_{E_8}} &= \chi_{\omega_4+\omega_8}^* - 4\chi_{\omega_1+\omega_6+\omega_8}^* - 2\chi_{\omega_2+\omega_3}^* \\ &\quad + \text{lower terms}, \\ F_{2A_3+2A_1, W_{E_8}} &= \chi_{\omega_4+\omega_8}^* - 4\chi_{\omega_1+\omega_6+\omega_8}^* - 4\chi_{\omega_2+\omega_3}^* \\ &\quad + \text{lower terms}. \end{aligned}$$

Considering the coefficients of the terms $\chi_{\omega_4+\omega_8}^*$ and $\chi_{\omega_2+\omega_3}^*$, we see that any non-trivial linear combination $c_1 F_{A_4+2A_1, W_{E_8}} + c_2 F_{2A_3+2A_1, W_{E_8}}$ is not a linear combination of the characters $\{F_{\Phi, W_{E_8}} : |\delta'_{\Phi}|^2 < |\delta'_{A_4+A_1}|^2 = 21\}$.

For the weight $2\omega_5$ and sub-root systems $A_5 + A_2 + A_1$, $2A_4$, suppose some non-trivial linear combination $c_1 F_{A_5+A_2+A_1, W_{E_8}} + c_2 F_{2A_4, W_{E_8}}$ is a linear combination of the characters $\{F_{\Phi, W_{E_8}} : |\delta'_{\Phi}|^2 < |2\omega_5|^2 = 40\}$. The sub-root systems with $33 \leq |2\delta'_{\Phi}|^2 \leq 39$ include $\{A_5 + A_2, D_4 + A_3, A_5 + 2A_1, A_5 + A_1, (A_5 + A_1)', A_5\}$, so there exists constants $c_3, c_4, c_5, c_6, c_7, c_8$ such that

$$\begin{aligned} &c_1 f_{A_5+A_2+A_1, E_8} + c_2 f_{2A_4, E_8} + c_3 f_{A_5+A_2, E_8} + c_4 f_{D_4+A_3, E_8} + \\ &c_5 f_{A_5+2A_1, E_8} + c_6 f_{A_5+A_1, E_8} + c_7 f_{(A_5+A_1)', E_8} + c_8 f_{A_5, E_8} \end{aligned}$$

is a polynomial of degree ≤ 32 .

By formulas in Subsection 9.3, we have

$$\begin{aligned} f_{A_5+A_2+A_1}(t) &= -t^{40} + 8t^{39} - 23t^{38} + 19t^{37} + 38t^{36} - 90t^{35} + 39t^{34} + \text{lower terms}, \\ f_{2A_4}(t) &= t^{40} - 8t^{39} + 22t^{38} - 12t^{37} - 53t^{36} + 88t^{35} + 2t^{34} + \text{lower terms}, \\ f_{D_4+A_3}(t) &= t^{38} - 7t^{37} + 16t^{36} - 4t^{35} - 33t^{34} + \text{lower terms}, \\ f_{A_5+A_1}(t) &= f_{(A_5+A_1)' }(t) = t^{36} - 6t^{35} + 11t^{34} + \text{lower terms}. \end{aligned}$$

Moreover,

$$f_{A_5+A_2}(t) = t^{39} + \text{lower terms},$$

$$\begin{aligned} f_{A_5+2A_1}(t) &= -t^{37} + \text{lower terms}, \\ f_{A_5}(t) &= -t^{35} + \text{lower terms}. \end{aligned}$$

Considering the coefficients of the terms t^{40} and t^{39} , we get

$$-c_1 + c_2 = 0, \quad 8c_1 - 8c_2 + c_3 = 0.$$

Thus $c_2 = c_1$ and $c_3 = 0$. Considering the coefficients of the terms t^{38} , t^{37} , we get

$$-23c_1 + 22c_2 + c_4 = 0, \quad 19c_1 - 12c_2 - 7c_4 - c_5 = 0.$$

Thus $c_4 = c_1$ and $c_5 = 0$. Considering the coefficients of t^{36} , t^{35} , we get

$$38c_1 - 53c_2 + 16c_4 + (c_6 + c_7) = 0$$

and

$$-90c_1 + 88c_2 - 4c_4 - 6(c_6 + c_7) - c_8 = 0.$$

Hence $c_6 + c_7 = -c_1$ and $c_8 = 0$. Finally, considering the coefficient of t^{34} , we get

$$0 = 39c_1 + 2c_2 - 33c_3 + 11(c_6 + c_7) = -3c_1.$$

Therefore $c_1 = c_2 = 0$.

For the weight $2\omega_1 + 2\omega_6$ and sub-root systems A_6 , $2D_4$, since $\Psi = E_8$ has no sub-root systems Φ with $e(\Phi) = 55$, we can show any non-trivial linear combination

$$c_1 F_{A_6, W_{E_8}} + c_2 F_{2D_4, W_{E_8}}$$

is not a linear combination of the characters $\{F_{\Phi, W_{E_8}} : |\delta'_{\Phi}|^2 < |2\omega_1 + 2\omega_6|^2 = 56\}$.

This proves the conclusion in the E_8 case.

Type F_4 . As in the proof of Theorem 8.14, we have observed that those weights appearing more than once in $\{2\delta'_{\Phi} \mid \Phi \subset F_4\}$ and the sub-root systems for which they appeared in are as follows:

- (1) ω_1 : A_1^L , $2A_1^S$, appears 2 times.
- (2) ω_3 : $A_1^L + A_1^S$, $3A_1^S$, appears 2 times.
- (3) $2\omega_4$: A_2^S , $2A_1^L$, $A_1^L + 2A_1^S$, $4A_1^S$, appears 4 times.
- (4) ω_2 : $3A_1^L$, $2A_1^L + 2A_1^S$, $A_1^L + A_2^S$, appears 3 times.
- (5) $2\omega_1$: A_2^L , $4A_1^L$, appears 2 times.
- (6) $\omega_1 + 2\omega_4$: A_3^S , B_2 , appears 2 times.
- (7) $2\omega_3$: $A_2^L + A_2^S$, $A_1^L + B_2$, $2A_1^S + B_2$, $A_1^L + A_3^S$, appears 4 times.
- (8) $2\omega_1 + 2\omega_4$: A_3^L , $2B_2$, appears 2 times.
- (9) $2\omega_3 + 2\omega_4$: D_4^S , C_3 , appears 2 times.

Since $C_4 \subset F_4$, $B_4 \subset F_4$, using the following equalities

$$\begin{aligned} c_1 + d_2 &= 2a_2, & c_1a_2 + d_2a_2 &= a_2^2, & c_1d_2 + d_2^2 &= 2a_2d_2, \\ c_1^2 + c_1d_2 &= 2a_2c_1, & a_3 &= c_1d_2, & c_1^3 + c_1^2d_2 &= 2a_2c_1^2, \\ c_1a_3 &= c_1^2d_2, & a_3 + d_2^2 &= 2a_2d_2, & c_1c_2 + d_2c_2 &= 2a_2c_2, \end{aligned}$$

and by the second and the third observations, we get

$$(38) \quad F_{A_1^L, W_{F_4}} + F_{2A_1^S, W_{F_4}} - 2F_{A_1^S, W_{F_4}} = 0,$$

$$(39) \quad F_{A_1^L + A_1^S, W_{F_4}} + F_{3A_1^S, W_{F_4}} - 2F_{2A_1^S, W_{F_4}} = 0,$$

$$(40) \quad F_{A_1^L + 2A_1^S, W_{F_4}} + F_{4A_1^S, W_{F_4}} - 2F_{3A_1^S, W_{F_4}} = 0,$$

$$(41) \quad F_{2A_1^L, W_{F_4}} + F_{A_1^L + 2A_1^S, W_{F_4}} - 2F_{A_1^L + A_1^S, W_{F_4}} = 0,$$

$$(42) \quad F_{A_2^S, W_{F_4}} - F_{A_1^L + 2A_1^S, W_{F_4}} = 0,$$

$$(43) \quad F_{3A_1^L, W_{F_4}} + F_{2A_1^L + 2A_1^S, W_{F_4}} - 2F_{2A_1^L + A_1^S, W_{F_4}} = 0,$$

$$(44) \quad F_{A_1^L + A_2^S, W_{F_4}} - F_{2A_1^L + 2A_1^S, W_{F_4}} = 0,$$

$$(45) \quad F_{A_2^L, W_{F_4}} + F_{4A_1^L, W_{F_4}} - 2F_{3A_1^L, W_{F_4}} = 0,$$

]

$$(46) \quad F_{A_1^L + B_2, W_{F_4}} + F_{2A_1^S + B_2, W_{F_4}} - 2F_{A_1^S + B_2, W_{F_4}} = 0.$$

Moreover, we will prove three more equalities

$$(47) \quad F_{A_3^S, W_{F_4}} - F_{B_2, W_{F_4}} + 2F_{A_2^L + A_1^S, W_{F_4}} - 2F_{2A_1^L + 2A_1^S, W_{F_4}} = 0,$$

$$(48) \quad F_{A_2^L + A_2^S, W_{F_4}} + F_{A_1^L + B_2, W_{F_4}} - F_{A_1^S + B_2, W_{F_4}} + 2F_{B_2, W_{F_4}} - 3F_{A_2^L + A_1^S, W_{F_4}} = 0,$$

$$(49) \quad F_{A_1^L + A_3^S, W_{F_4}} - F_{A_1^L + B_2, W_{F_4}} + 2F_{A_1^S + B_2, W_{F_4}} - 2F_{A_3^S, W_{F_4}} = 0.$$

We will show these relations generate all linear relations among $\{F_{\Phi, W_{F_8}} \mid \Phi \subset F_8\}$.

Lemma 9.19. *Given an integer k with $1 \leq k \leq 12$ and $k \neq 9$, there exists a unique dominant integral weight λ for the root system $\Psi = F_4$ such that $|\lambda|^2 = k$.*

Given $k = 9$, there exist two dominant integral weights λ for the root system $\Psi = F_4$ such that $|\lambda|^2 = k$. They are $3\omega_4$ and $\omega_1 + \omega_3$.

Proof. The inverse to the Cartan matrix of F_4 is $\begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}$, so for a dominant

integral weight $\lambda = \sum_{1 \leq i \leq 4} a_i \omega_i$ ($a_i \in \mathbb{Z}_{\geq 0}$), we have

$$|\lambda|^2 = 2a_1^2 + 6a_2^2 + 3a_3^2 + a_4^2 + (6a_1a_2 + 4a_1a_3 + 2a_1a_4 + 8a_2a_3 + 4a_2a_4 + 3a_3a_4).$$

Suppose $|\lambda|^2 \leq 12$. First we must have $a_2 = 0, 1$. If $a_2 = 1$, we have

$$\lambda = \omega_2 \text{ (6), } \omega_2 + \omega_4 \text{ (11)}.$$

We also have $a_3 \leq 2$. When $a_3 \geq 1$, we have

$$\lambda = 2\omega_3 \text{ (12), } \omega_3 + \omega_1 \text{ (9), } \omega_3 + \omega_4 \text{ (7), } \omega_3 \text{ (3)}.$$

When $a_2 = a_3 = 0$, we have

$$\lambda = 2\omega_1 \text{ (8), } \omega_1 + 2\omega_4 \text{ (10), } \omega_1 + \omega_4 \text{ (5), } \omega_1 \text{ (2), } 3\omega_4 \text{ (9), } 2\omega_4 \text{ (4), } \omega_4 \text{ (1)}.$$

This proves the lemma. \square

Proof of the three equalities. First any term χ_λ^* appearing in these three equalities has $|\lambda|^2 \leq 12$. By Lemma 9.19, to prove these equalities, we just need to prove the corresponding equalities about generating functions f_{Φ, F_4} and to calculate the coefficients of the terms χ_λ^* with $|\lambda|^2 = 9$.

For the functions f_{Φ, F_4} , we have

$$\begin{aligned} & f_{A_3^S, F_4} - f_{B_2, F_4} + 2f_{A_2^L + A_1^S, F_4} - 2f_{2A_1^L + 2A_1^S, F_4} \\ &= (1-t)^3(1-t^2)^2(1-t^3) - (1-t)(1-t^2)(1-t^3)(1-t^4) \\ & \quad + 2(1-t)(1-t^2)^2(1-t^4) - 2(1-t)^2(1-t^2)^2 \\ &= (1-t)(1-t^2)(1-t^3)(-2t+2t^3) + 2(1-t)(1-t^2)^2(t-t^4) \\ &= 0, \end{aligned}$$

$$\begin{aligned}
& f_{A_2^L+A_2^S, F_4} + f_{A_1^L+B_2, F_4} - f_{A_1^S+B_2, F_4} + 2f_{B_2, F_4} - 3f_{A_2^L+A_1^S, F_4} \\
&= (1-t)^2(1-t^2)^3(1-t^4) + (1-t)(1-t^2)^2(1-t^3)(1-t^4) \\
&\quad - (1-t)^2(1-t^2)(1-t^3)(1-t^4) + 2(1-t)(1-t^2)(1-t^3)(1-t^4) \\
&\quad - 3(1-t)(1-t^2)^2(1-t^4) \\
&= (1-t)(1-t^2)^2(1-t^4)(2-t-t^2) + (1-t^2)^2(1-t^3)(1-t^4) \\
&\quad - 3(1-t)(1-t^2)^2(1-t^4) \\
&= (1-t^2)^2(1-t^4)(3-3t) - 3(1-t)(1-t^2)^2(1-t^4) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
& f_{A_1^L+A_3^S, F_4} - f_{A_1^L+B_2, F_4} + 2f_{A_1^S+B_2, F_4} - 2f_{A_3^S, F_4} \\
&= (1-t)^3(1-t^2)^3(1-t^3) - (1-t)(1-t^2)^2(1-t^3)(1-t^4) \\
&\quad + 2(1-t)^2(1-t^2)(1-t^3)(1-t^4) - 2(1-t)^3(1-t^2)^2(1-t^3) \\
&= (1-t)(1-t^2)^2(1-t^3)(-2t+2t^3) \\
&\quad + 2(1-t)^2(1-t^2)(1-t^3)(t+t^2-t^3-t^4) \\
&= 0.
\end{aligned}$$

The terms χ_λ^* with $|\lambda|^2 = 9$ in

$$\begin{aligned}
& F_{A_3^S, W_{F_4}} - F_{B_2, W_{F_4}} + 2F_{A_2^L+A_1^S, W_{F_4}} - 2F_{A_1^L+2A_1^S, W_{F_4}}, \\
& F_{A_2^L+A_2^S, W_{F_4}} + F_{A_1^L+B_2, W_{F_4}} - F_{A_1^S+B_2, W_{F_4}} + 2F_{B_2, W_{F_4}} - 3F_{A_2^L+A_1^S, W_{F_4}}, \\
& F_{A_1^L+A_3^S, W_{F_4}} - F_{A_1^L+B_2, W_{F_4}} + 2F_{A_1^S+B_2, W_{F_4}} - 2F_{A_3^S, W_{F_4}}
\end{aligned}$$

are

$$\begin{aligned}
& (-2\chi_{\omega_1+\omega_3}^* - \chi_{3\omega_4}^*) - (-\chi_{3\omega_4}^*) + 2\chi_{\omega_1+\omega_3}^* = 0, \\
& (4\chi_{\omega_1+\omega_3}^* + 2\chi_{3\omega_4}^*) + (-\chi_{3\omega_4}^*) - (\chi_{\omega_1+\omega_3}^* - \chi_{3\omega_4}^*) + 2(-\chi_{3\omega_4}^*) - 3\chi_{\omega_1+\omega_3}^* = 0, \\
& (-6\chi_{\omega_1+\omega_3}^* - \chi_{3\omega_4}^*) - (-\chi_{3\omega_4}^*) + 2(\chi_{\omega_1+\omega_3}^* - \chi_{3\omega_4}^*) - 2(-2\chi_{\omega_1+\omega_3}^* - \chi_{3\omega_4}^*) = 0
\end{aligned}$$

respectively.

Therefore we get the three equalities. \square

For the weights $2\omega_1 + 2\omega_4$ and $2\omega_3 + 2\omega_4$, since $\Psi = F_4$ has no sub-root systems Φ with $e(\Phi) = 19$ or $e(\Phi) = 27$, we can show any non-trivial linear combination $c_1 F_{A_3^L, W_{F_4}} + c_2 F_{B_2, W_{F_4}}$ is not a linear combination of the characters $\{F_{\Phi, W_{F_4}} : |\delta'_\Phi|^2 < |\delta'_{A_4+A_1}|^2 = 20\}$ and any non-trivial linear combination $c_1 F_{D_4^S, W_{F_4}} + c_2 F_{C_3, W_{F_4}}$ is not a linear combination of the characters $\{F_{\Phi, W_{F_4}} : |\delta'_\Phi|^2 < |\delta'_{A_4+A_1}|^2 = 28\}$. Thus the relations as listed above generate all linear relations among $\{F_{\Phi, W_{F_4}} \mid \Phi \subset F_4\}$.

Type G_2 . For $\Psi = G_2$, the only non-conjugate reduced sub-root systems with conjugate leading terms are A_2^S and $A_1^L + A_1^S$. We have

$$(50) \quad F_{A_2^S, W_{G_2}} + F_{A_1^L+A_1^S, W_{G_2}} + F_{A_1^L, W_{G_2}} - 3F_{A_1^S, W_{G_2}} = 0.$$

This is the unique linear relation between the characters $\{F_{\Phi, W_{G_2}} \mid \Phi \subset G_2\}$.

Remark 9.20. Given an irreducible root system Ψ , if $\Psi_0 \not\cong \text{BC}_n$, then the characters $\{F_{\Phi, W_\Psi} : \Phi \subset \Psi_0, \text{rank } \Phi = \Psi_0\}$ are linearly independent. If $\Psi = \text{BC}_n$, for two sub-root systems Φ_1, Φ_2 of Ψ with $\text{rank } \Phi_1 = \text{rank } \Phi_2 = n$, $F_{\Phi_1, W_n} = F_{\Phi_2, W_n}$ if and only if $\Phi_1 \sim_{W_n} \Phi_2$. These follow from the results we showed above. Note that, these were also proved by Larsen-Pink. Actually this is the essential part to the proof of Theorem 1 in [LP]. On the other hand, Larsen-Pink have proved the existence of algebraic relations among

the polynomials $\{b_n, c_n, d_{n+1} | n \geq 1\}$. And they used this to construct non-conjugate closed subgroups with equal dimension data.

Proposition 9.21. *Given a compact connective Lie group G of type E_6 , E_7 , E_8 or F_4 , there exist non-isomorphic closed connected full rank subgroups with linearly dependent dimension data.*

Proof. In the case that G is of type E_6 , E_7 , E_8 , G possesses a Levi subgroup of type D_4 . Hence the conclusion follows from Proposition 9.11. In the case that G is of type F_4 , G possesses a subgroup isomorphic to $\text{Spin}(8)$. Therefore the conclusion follows from Proposition 9.11. \square

10. COMPARISON OF QUESTION 1.2 AND QUESTION 5.2

In this section we give constructions which show that Question 5.2 (or 5.1) is not an excessive generalization of Question 1.2 (or 1.1), as we promised after introducing Questions 5.1 and 5.2. The significance of these constructions is that each equality (or linear relation) we found in Question 5.1 (or 5.2) indeed corresponds to an equality (or a linear relation) of dimension data of closed connected subgroups in a suitable group. Another consequence of these constructions is showing that the group Γ° could be quite arbitrary.

Precisely, what we do is as follows. Given a root system Ψ' and a finite group W acting faithfully on Ψ' and containing $W_{\Psi'}$, we construct some pair (G, T) with G a compact Lie group and T a closed connected torus in G such that:

- (1) $\text{rank } \Psi_T = \dim T = \text{rank } \Psi'$.
- (2) $\Psi' \subset \Psi_T$ and is stable under Γ° .
- (3) $\Gamma^\circ = W$ as groups acting on Ψ' .
- (4) For each reduced sub-root system Φ of Ψ' , there exists a connected closed subgroup H of G with T a maximal torus of H and with root system $\Phi(H, T) = \Phi$.

Sub-root systems of any given irreducible root system Ψ_0 are classified in [Os], in Section 6 we have discussed this classification in the case that Ψ_0 is a classical irreducible root system or an exceptional irreducible root system of type F_4 or G_2 .

Proposition 10.1. *Given an irreducible root system Ψ_0 , there exists a compact connected simple Lie group G and a closed connected torus T in G such that:*

- (1) $\text{rank } \Psi_T = \dim T = \text{rank } \Psi_0$.
- (2) $\Psi_0 \subset \Psi_T$ and is stable under Γ° .
- (3) $\Gamma^\circ = W_{\Psi_0}$ as groups acting on Ψ_0 .
- (4) For each reduced sub-root system Φ of Ψ_0 , there exists a connected closed subgroup H of G with T a maximal torus of H and with root system $\Phi(H, T) = \Phi$.

Proof. Simply laced case. In the case that Ψ_0 is a simply laced irreducible root system, let \mathfrak{u}_0 be a compact simple Lie algebra with root system isomorphic to Ψ_0 . Taking $G = \text{Int}(\mathfrak{u}_0)$ and T a maximal torus of G , the conclusion in the proposition is satisfied for (G, T) .

Type C_n . In the case that $\Psi_0 = C_n$, let $G = \text{SU}(2n)$,

$$T = \{\text{diag}\{z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}\} \mid |z_1| = |z_2| = \dots = |z_n| = 1\}$$

and $\theta = \text{Ad} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$. Then θ is an involutive automorphism of G and $\text{Lie } T$ is a maximal abelian subspace of $\mathfrak{p}_0 = \{X \in \text{Lie } G \mid \theta(X) = -X\}$. In this example, the restricted root system $\Phi(G, T) \cong C_n$ (cf. [Kn], Page 424), each long root occurs with

multiplicity one, each short root occurs with multiplicity two and $C_G(T)$ is the subgroup of diagonal matrices in G . We have

$$\Gamma^\circ = N_G(T)/C_G(T) = \{-1\}^n \rtimes S_n = W_{C_n}.$$

Denote by

$$J_k = \begin{pmatrix} \frac{1}{\sqrt{2}}I_k & & \frac{1}{\sqrt{2}}iI_k \\ & I_{n-k} & \\ \frac{1}{\sqrt{2}}iI_k & & \frac{1}{\sqrt{2}}I_k \\ & & & I_{n-k} \end{pmatrix},$$

which is an $2n \times 2n$ matrix. Define the subgroups

$$\begin{aligned} A_{k-1} &= \{ \text{diag}\{A, K, \overline{A}, \overline{K}\} \mid A \in U(k), K \in T_{n-k} \}, \\ C_k &= \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & K & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & \overline{K} \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(k), K \in T_{n-k} \right\}, \\ D'_k &= \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & K & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & \overline{K} \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SO}(2k), K \in T_{n-k} \right\} \end{aligned}$$

and

$$D_k = J_k D'_k J_k^{-1}.$$

Thus T is a maximal torus of D_k , C_k and A_{k-1} , and the root systems

$$\Phi(D_k, T) = D_k,$$

$$\Phi(C_k, T) = C_k$$

and

$$\Phi(A_{k-1}, T) = A_{k-1}.$$

By [Os], we know that any sub-root system $\Phi \subset C_n$ is of the form

$$A_{r_1-1} + \cdots + A_{r_i-1} + C_{s_1} + \cdots + C_{s_j} + D_{t_1} + \cdots + D_{t_k},$$

where $r_1, \dots, r_i, s_1, \dots, s_j, t_1, \dots, t_k \geq 1$ and

$$r_1 + \cdots + r_i + s_1 + \cdots + s_j + t_1 + \cdots + t_k = n.$$

Here we regard $A_0 = \emptyset$. Using subgroups of block-form, we see that each sub-root system Φ of Ψ_0 is of the form $\Phi(H, T)$ for some closed connected subgroup H of G with T a maximal torus of H .

Type BC_n . In the case that $\Psi_0 = \text{BC}_n$, let $G = \text{SU}(2n+t)$, $t \geq n$,

$$T = \{ \text{diag}\{I_t, z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1} : |z_1| = |z_2| = \cdots = |z_n| = 1\} \}$$

and $\theta = \text{Ad} \begin{pmatrix} I_t & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$. Then θ is an involutive automorphism of G and $\text{Lie } T$ is a

maximal abelian subspace of $\mathfrak{p}_0 = \{X \in \text{Lie } G \mid \theta(X) = -X\}$. Moreover, the restricted root system

$$\Phi(G, T) = \text{BC}_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{\pm e_i, \pm 2e_i \mid 1 \leq i \leq n\}$$

(cf. [Kn], Page 424), each root $2e_i$ occurs with multiplicity one, each root $\pm e_i \pm e_j$ occurs with multiplicity two and each root e_i occurs with multiplicity $2t$. We have

$$\Gamma^\circ = N_G(T)/C_G(T) = W_{\text{BC}_n} = \text{Aut}(\text{BC}_n) = \Gamma.$$

Denote by

$$J_k = \begin{pmatrix} I_t & & & \\ & \frac{1}{\sqrt{2}}I_k & 0 & 0\frac{1}{\sqrt{2}}iI_k & 0 \\ & 0 & I_{n-k} & 0 & 0 \\ & \frac{1}{\sqrt{2}}iI_k & 0 & \frac{1}{\sqrt{2}}I_k & 0 \\ & 0 & 0 & 0 & I_{n-k} \end{pmatrix},$$

which is a $2n + t$ square matrix. Define the subgroups

$$A_{k-1} = \left\{ \begin{pmatrix} I_t & & & \\ & A & 0 & 0 & 0 \\ & 0 & K & 0 & 0 \\ & 0 & 0 & \overline{A} & 0 \\ & 0 & 0 & 0 & \overline{K} \end{pmatrix} \mid A \in \mathrm{U}(k), K \in T_{n-k} \right\},$$

$$B'_k = \left\{ \begin{pmatrix} I_{t-1} & & & \\ & A & 0 & B & 0 \\ & 0 & K & 0 & 0 \\ & C & 0 & D & 0 \\ & 0 & 0 & 0 & \overline{K} \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2k+1), K \in T_{n-k} \right\},$$

$$C_k = \left\{ \begin{pmatrix} I_t & & & \\ & A & 0 & B & 0 \\ & 0 & K & 0 & 0 \\ & C & 0 & D & 0 \\ & 0 & 0 & 0 & \overline{K} \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(k), K \in T_{n-k} \right\},$$

$$D'_k = \left\{ \begin{pmatrix} I_t & & & \\ & A & 0 & B & 0 \\ & 0 & K & 0 & 0 \\ & C & 0 & D & 0 \\ & 0 & 0 & 0 & \overline{K} \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2k), K \in T_{n-k} \right\},$$

$$B_k = J_k B'_k J_k^{-1}$$

and

$$D_k = J_k D'_k J_k^{-1}.$$

Thus T is a maximal torus of D_k , C_k , A_{k-1} and B_k , and the root systems

$$\Phi(D_k, T) = D_k,$$

$$\Phi(C_k, T) = C_k,$$

$$\Phi(A_{k-1}, T) = A_{k-1}$$

and

$$\Phi(B_k, T) = B_k.$$

Using subgroups of block form, we see that each reduced root system Φ of Ψ_0 is of the form $\Phi(H, T)$ for some connected closed subgroup H of G with T a maximal torus of H . Note that the condition $t \geq n$ makes sure that the sub-root system nB_1 corresponds to a subgroup.

Type B_n In the case that $\Psi_0 = B_n$, let (G, T) be the same as the above for BC_n . Then this pair satisfies the desired conclusion.

Type F₄. In the case that $\Psi_0 = F_4$. Recall that the complex simple Lie algebra $\mathfrak{e}_7(\mathbb{C})$ has a real form with a restricted root system isomorphic to F_4 (cf. [Kn], Page 425). Let $G = \mathrm{Aut}(\mathfrak{e}_7)$ be the automorphism group of a compact simple Lie algebra of type E_7 . Then we can choose an involution θ_0 in G and a closed connected torus T in G such that $\theta|_T = -1$, $\mathrm{Lie} T$ is a maximal abelian subspace of $\mathfrak{e}_7^{-\theta} = \{X \in \mathfrak{e}_7 \mid \mathrm{Ad}(\theta)(X) = -X\}$ and

the restricted root system $\Phi(G, T) \cong F_4$. Write $\Psi_0 = \Phi(G, T)$. Then $\Psi_0 \cong F_4$ and we show that any sub-root system Φ of Ψ_0 is of the form $\Phi = \Phi(H, T)$ for a closed connected subgroup H of G with T a maximal torus of H . To show this, we use some results from [HY]. Starting with the Klein four subgroup Γ_6 of G (cf. [HY], Table 4), we have $G^{\Gamma_6} = (G^{\Gamma_6})_0 \times \Gamma$ and $(G^{\Gamma_6})_0 \cong F_4$. From [HY], we know that some involutions in $(G^{\Gamma_6})_0$ are conjugate to θ_0 in G . Without loss of generality we may assume that $\theta_0 \in (G^{\Gamma_6})_0$. Choosing any $1 \neq \theta_1 \in \Gamma_6$, we have (cf. [HY])

$$(G^{\theta_1})_0 \cong (E_6 \times U(1)) / \langle (c, e^{\frac{2\pi i}{3}}) \rangle,$$

$$(G^{\theta_0 \theta_1})_0 \cong SU(8) / \langle iI \rangle$$

and

$$\theta_0 \in (G^{\theta_1})_0 \cap (G^{\theta_0 \theta_1})_0.$$

Here $1 \neq c \in Z(E_6)$ with $o(c) = 3$. By these, we get three closed connected subgroups H_1, H_2, H_3 of G containing T and θ_0 , and such that

$$\text{Lie } H_1 \cong \mathfrak{su}(8),$$

$$\text{Lie } H_2 \cong \mathfrak{e}_6$$

and

$$\text{Lie } H_3 \cong \mathfrak{f}_4.$$

Moreover, $\text{Lie } T$ is a maximal abelian subspace of each of $(\text{Lie } H_i)^{-\theta_0}$, $i = 1, 2, 3$. By the classification of sub-root systems of F_4 (cf. Section 6), we can show that any sub-root system Φ of $\Psi_0 \cong F_4$ is of the form $\Phi(H, T)$ for a connected closed subgroup H contained in one of H_1, H_2, H_3 .

Type G_2 . In the case that $\Psi_0 = G_2$, let $G = \text{Spin}(8)$. There exist (cf. [He], Page 517) two order three outer automorphisms θ, θ' of G such that

$$H_1 = G^\theta \cong G_2,$$

$$H_2 = G^{\theta'} \cong \text{PSU}(3)$$

and H_1, H_2 share a common maximal torus T . Thus $\Phi(G, T) \cong G_2$, each long root occurs with multiplicity 1 and each short root occurs with multiplicity 3. By the classification of sub-root systems of F_4 (cf. Section 6), one can show that any sub-root system Φ of Ψ_0 is of the form $\Phi(H, T)$ for some connected closed subgroup H of H_1 or H_2 . \square

Remark 10.2. When Ψ_0 is an irreducible root system not of type B_n ($n \geq 3$), in the above construction we actually have $\Psi'_T = \Psi_0$.

Proposition 10.3. Given a root system Ψ' and a finite group W between $W_{\Psi'}$ and $\text{Aut}(\Psi')$, there exists a compact (not necessarily connected) Lie group G with a bi-invariant Riemannian metric m and a connected closed torus T in G such that:

- (1) $\text{rank } \Psi_T = \dim T = \text{rank } \Psi'$.
- (2) $\Psi' \subset \Psi_T$ and is stable under Γ° .
- (3) $\Gamma^\circ = W$ as groups acting on Ψ' .
- (4) For each reduced sub-root system Φ of Ψ' , there exists a connected closed subgroup H of G with T a maximal torus of H and with root system $\Phi(H, T) = \Phi$.

Proof. Decomposing Ψ' into a disjoint union of (orthogonal) simple root systems and joining those isomorphic simple factors, we may write Ψ' as the form

$$\Psi' = \Psi'_1 \sqcup \cdots \sqcup \Psi_s$$

where each $\Psi'_i = m_i \Psi_{i,0}$ is a union of m_i root systems all isomorphic to an irreducible root system $\Psi_{i,0}$.

For each i , by Proposition 10.1, we get (G_i, T_i) corresponds to the irreducible root system $\Psi_{i,0}$ and the finite group $W_{\Psi_{i,0}}$ satisfying all conditions in the Proposition. Let

$$G' = G_1^{m_1} \times \cdots \times G_s^{m_s}$$

and

$$T = T_1^{m_1} \times \cdots \times T_s^{m_s}.$$

Then (G', T) corresponds to the root system Ψ' and the finite group $W_{\Psi'}$ with all conditions in the conclusion satisfied.

Among $\{\Psi_{i,0} \mid 1 \leq i \leq s\}$, we may assume that $\text{Aut}(\Psi_{i,0}) \neq W_{\Psi_{i,0}}$ happens exactly when $1 \leq i \leq t$. For each $i \leq t$, since $\text{Aut}(\Psi_{i,0}) \neq W_{\Psi_{i,0}}$, $\Psi_{i,0}$ must be simply laced. In this case we have $G_i = \text{Int}(\mathfrak{u}_i)$ for a compact simple Lie algebra \mathfrak{u}_i with root system $\Psi_{i,0}$. Let

$$G'' = (\text{Aut}(\mathfrak{u}_1)^{m_1} \rtimes S_{m_1}) \times \cdots \times (\text{Aut}(\mathfrak{u}_t)^{m_t} \rtimes S_{m_t}) \times (G_{t+1}^{m_{t+1}} \rtimes S_{m_{t+1}}) \times \cdots \times (G_s^{m_s} \rtimes S_{m_s})$$

and

$$T = T_1^{m_1} \times \cdots \times T_s^{m_s}.$$

Hence (G'', T) corresponds to the root system Ψ' and the finite group $\text{Aut}(\Psi')$ with all conditions in the conclusion satisfied.

We have $G''/G' \cong \text{Aut}(\Psi')/W_{\Psi'}$. Corresponding to the subgroup $W/W_{\Psi'}$ of $\text{Aut}(\Psi')/W_{\Psi'}$, we get a subgroup G of G'' containing G' and with $G/G' = W/W_{\Psi'}$ in the identification $G''/G' = \text{Aut}(\Psi')/W_{\Psi'}$. Therefore (G, T) corresponds to (Ψ', W) with all conditions in the conclusion satisfied. \square

As in the proof of Proposition 10.1, given an irreducible root system $\Psi_0 = \text{BC}_n$, let $G_1 = \text{SU}(2n+t)$ for $t \geq n$ and

$$T_1 = \{\text{diag}\{z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}, \underbrace{1, \dots, 1}_t\} \mid |z_1| = |z_2| = \cdots = |z_n| = 1\}.$$

Denote by S_1 the group of diagonal matrices in G_1 . It is a maximal torus of G_1 . Let

$$\epsilon_i(\text{diag}\{z_1, \dots, z_{2n+t}\}) = z_i$$

for any $\text{diag}\{z_1, \dots, z_{2n+t}\} \in S_1$ and

$$e_i(\text{diag}\{z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}, \underbrace{1, \dots, 1}_t\}) = z_i$$

for any $\text{diag}\{z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}, \underbrace{1, \dots, 1}_t\} \in T$. Thus

$$\epsilon_i|_T = \begin{cases} e_i & \text{if } 1 \leq i \leq n \\ e_{i-n}^{-1} & \text{if } n+1 \leq i \leq 2n \\ 1 & \text{if } 2n+1 \leq i \leq 2n+t. \end{cases}$$

The following example is a modification of an example in [LP], Page 392.

Example 10.4. Given $r \geq n$, denote by $G_1 = \text{SU}(2n+t)$, $T_1 \subset G_1$ the subgroup of diagonal matrices, $G_2 = (G_1)^r$ and $T_2 = (T_1)^r$. Write

$$\lambda_j = \sum_{1 \leq i \leq n} (nj - i + 1)\epsilon_i$$

for $1 \leq j \leq r$. Let $V_j = V_{\lambda_j}$ be an irreducible representation of G_1 with highest weight λ_j and

$$V = \bigoplus_{\sigma \in A_r} V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(r)}.$$

Denote by $G = \mathrm{SU}(k)$, where

$$k = \frac{r!}{2} \prod_{1 \leq j \leq r} \dim V_j.$$

The representation V gives us an embedding $G_2 \subset G$. Write T for the image of T_2 under this embedding. Since $G = \mathrm{SU}(k)$ is simple, a biinvariant Riemannian metric m on G is unique up to a scalar multiple. Hence Ψ_T does not depend on the choice of m . In this example, we have:

- (1) $\mathrm{rank} \Psi_T = \dim T = rn$.
- (2) $\Psi' = r \mathrm{BC}_n \subset \Psi_T$ and it is stable under Γ° .
- (3) $\Gamma^\circ = W_{\Psi'} \rtimes A_r = (W_{\mathrm{BC}_n})^r \rtimes A_r$ as groups acting on Ψ' .
- (4) For each reduced sub-root system Φ of Ψ' , there exists a closed connected subgroup H of G with T a maximal torus of H and with root system $\Phi(H, T) = \Phi$.

Proof. First we have $(W_{\mathrm{BC}_n})^r \rtimes A_r \subset \Gamma^\circ$ and $\Psi_T = \mathrm{BC}_{rn}$ by our construction of the pair (G, T) .

We have $\Psi' = r \mathrm{BC}_n \subset \Psi_T$ since $r \mathrm{BC}_n$ is the root system of (G_2, T) . By Proposition 10.1, any reduced sub-root system Φ of $r \mathrm{BC}_n$ is of the form $\Phi(H, T)$ for a closed connected subgroup H of G_2 .

The equality $\Gamma^\circ = (W_{\mathrm{BC}_n})^r \rtimes A_r$ can be proven using the idea in [LP], Page 392. It proceeds as follows. By the construction of G_2 and V , the character

$$\chi_{V_1 \otimes \cdots \otimes V_r}|_T$$

has $\lambda = (nr, nr - 1, \dots, 1)$ as a leading term. It is regular with respect to $\mathrm{Aut}(T, m) = W_{\mathrm{BC}_{rn}}$. The weights appearing in $\chi_V|_T$ of maximal length are in the orbit $((W_{\mathrm{BC}_n})^r \rtimes A_r)\lambda$. Hence $\Gamma^\circ = (W_{\mathrm{BC}_n})^r \rtimes A_r$. \square

Remark 10.5. In Example 10.4, moreover we have $\Psi'_T = r \mathrm{BC}_n = \Psi'$ since otherwise Γ° must be larger than $(W_{\mathrm{BC}_n})^r \rtimes A_r$.

Proposition 10.6. Given a root system Ψ' and a finite group W between $W_{\Psi'}$ and $\mathrm{Aut}(\Psi')$, there exists some $G = \mathrm{SU}(k)$ and a closed connected torus T in G such that:

- (1) $\mathrm{rank} \Psi_T = \dim T = \mathrm{rank} \Psi'$.
- (2) $\Psi' \subset \Psi_T$ and it is stable under Γ° .
- (3) $\Gamma^\circ = W$ as groups acting on Ψ' .
- (4) For each reduced sub-root system Φ of Ψ' , there exists a closed connected subgroup H of G with T a maximal torus of H and with root system $\Phi(H, T) = \Phi$.

Proof. As in the proof of Proposition 10.3, we write Ψ' as the form

$$\Psi' = \Psi'_1 \bigsqcup \cdots \bigsqcup \Psi'_s,$$

where each $\Psi'_i = m_i \Psi_{i,0}$ is a union of m_i root systems all isomorphic to an irreducible root system $\Psi_{i,0}$.

For each i , by Proposition 10.1, we get (G_i, T_i) corresponds to the root system $\Psi_{i,0}$ and the finite group $W_{\Psi_{i,0}}$ satisfying all conditions in the Proposition. Moreover, if $\Psi_{i,0}$ is simply laced, then $G_i = \mathrm{Int}(\mathfrak{u}_i)$ for a compact simple Lie algebra \mathfrak{u}_i with root system isomorphic to $\Psi_{i,0}$. Let

$$G' = G_1^{m_1} \times \cdots \times G_s^{m_s}$$

and

$$T' = T_1^{m_1} \times \cdots \times T_s^{m_s}.$$

Among $\Psi_{i,0}$, we assume that $\text{Aut}(\Psi_{i,0}) \neq W_{\Psi_{i,0}}$ happens exactly when $1 \leq i \leq t$. We have

$$\text{Aut}(\Psi')/W_{\Psi'} = (\text{Out}(\Psi_{1,0})^{m_1} \rtimes S_{m_1}) \times \cdots \times (\text{Out}(\Psi_{t,0})^{m_t} \rtimes S_{m_t}) \times S_{m_{t+1}} \times \cdots \times S_{m_s}.$$

Here $\text{Out}(\Psi_{i,0}) = \text{Aut}(\Psi_{i,0})/W_{\Psi_{i,0}}$. The group $\text{Aut}(\Psi')/W_{\Psi'}$ acts on $\text{Rep}(G')$ (here $G_i = \text{Int}(\mathfrak{u}_i)$ plays a role) through its action on the dominant integral weights.

For any $1 \leq i \leq s$, choose a maximal torus S_i of G_i containing T_i . If $i \leq t$, we choose m_i dominant integral weights $\lambda_{i,1}, \dots, \lambda_{i,m_i}$ of $S_i = T_i$ such that the set $\{\gamma\lambda_{i,j} | \gamma \in \text{Aut}(\Psi_{i,0}), 1 \leq j \leq m_i\}$ has cardinality exactly $m_i |\text{Aut}(\Psi_{i,0})|$. If $i \geq t+1$, the restriction map of weight lattices

$$p_i : \hat{S}_i \longrightarrow \hat{T}_i$$

is surjective and it is an orthogonal projection. Choose m_i regular dominant integral weights $\lambda_{i,1}, \dots, \lambda_{i,m_i}$ of S_i with an additional property: each $\lambda_{i,j}$ is orthogonal to the weights in $\ker p_i$ and their images under p_i are regular and distinct to each other. Thus the weights of maximal length in $V_{\lambda_{i,j}}|_{T_i}$ are those in the orbit $W_{\Psi_{i,0}}\lambda_{i,j}$, and each occurs with multiplicity one. Here $V_{\lambda_{i,j}}$ is an irreducible representation of G_i with highest weight $\lambda_{i,j}$. Denote by

$$V = \bigoplus_{\sigma \in W/W_{\Psi'}} \sigma(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}).$$

and $G = \text{SU}(k)$, where

$$k = \dim V = \frac{|W|}{|W_{\Psi'}|} \prod_{1 \leq j \leq r} \dim V_j.$$

The representation V gives us an embedding $G' \subset G = \text{SU}(k)$. Write T for the image of T' under this embedding. Arguing similarly as in the proof of Example 10.4, we get $\Gamma^\circ = W$. By Proposition 10.1, the other conditions in the conclusion are also satisfied. \square

Remark 10.7. In the above construction, we know $\Psi'_T \supset \Psi'$ by Property (4) of Proposition 10.6 and $W_{\Psi'_T} \subset \Gamma^\circ$ by Proposition 3.9. From this Ψ'_T is almost determined, which should be close to being equal to Ψ' . On the other hand, Ψ_T is probably much larger than Ψ' . For example it may happen that $\Psi' = r \text{BC}_n$ and $\Psi'_T = \text{BC}_{nr}$.

Can we make an example with $\Psi'_T = \Psi'$ in Proposition 10.6? Moreover, can we make an example with $\Psi_T = \Psi'$?

11. IRREDUCIBLE SUBGROUPS

Let $G = \text{U}(n)$. In this section, we study dimension data of closed subgroups H of G acting irreducibly on \mathbb{C}^n .

Choose a prime p , an integer $m \geq 4$ and a prime $q > p^m$. Let $n = p^m$, $T \subset G$ be the subgroup of diagonal unitary matrices and

$$A = \{\text{diag}\{a_1, a_2, \dots, a_n\} | a_1^q = a_2^q = \cdots = a_n^q = 1\}.$$

Lemma 11.1. $C_G(A) = T$ and $N_G(A) = T \rtimes S_n$, where S_n is the subgroup of permutation matrices in G .

There exists a unique conjugacy class of subgroups \overline{N} of S_n isomorphic to $(C_p)^m$ and with non-identity elements all conjugate to $(1, 2, \dots, p)(p+1, p+2, \dots, 2p) \cdots (n-p+1, n-p+2, \dots, n)$.

For any subgroup \overline{N} of S_n as above, $C_{S_n}(\overline{N}) = \overline{N}$ and

$$N_{S_n}(\overline{N})/\overline{N} \cong \text{GL}(m, \mathbb{F}_p).$$

Proof. The first statement is clear. For the second statement, one can prove the uniqueness by induction on m . Again, one can show $C_{S_n}(\overline{N}) = \overline{N}$ by induction on m . Finally, by the uniqueness of \overline{N} , we get

$$N_{S_n}(\overline{N})/\overline{N} \cong \text{Aut}(\overline{N}) \cong GL(m, \mathbb{F}_p).$$

□

Now we specify $p = 2$ and $m = 4$. Let $\overline{H} = N_{S_n}(\overline{N})$. Then $\overline{H}/\overline{N} \cong GL(4, \mathbb{F}_2)$. Choose a subgroup \overline{H}_1 of \overline{H} with $\overline{H}_1/\overline{N}$ corresponding to

$$\left\langle \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

and a subgroup \overline{H}_2 of \overline{H} with $\overline{H}_2/\overline{N}$ corresponding to

$$\left\langle \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

Let $K = A \rtimes S_n$ and $N \subset K$ with $N/A = \overline{N}$. Write $H = N_K(N)$. Then $H/N = \overline{H}$. Let $H_1, H_2 \subset H$ with $H_1/N = \overline{H}_1$ and $H_2/N = \overline{H}_2$.

Proposition 11.2. *The two subgroups H_1 and H_2 act irreducibly on \mathbb{C}^{16} , have the same dimension data and are non-isomorphic.*

Proof. Since N acts irreducibly on \mathbb{C}^{16} and H_1, H_2 contain N , H_1 and H_2 act irreducibly on \mathbb{C}^{16} .

To show H_1 and H_2 having the same dimension data in G , it is sufficient to show they have the same dimension data in H . Since N is a normal subgroup of H , H_1 and H_2 , it is sufficient to show $\overline{H}_1 = H_1/N$ and $\overline{H}_2 = H_2/N$ have the same dimension data in $\overline{H} = H/N \cong GL(4, \mathbb{F}_2)$. We have $\overline{H}_1 \cong \overline{H}_2 \cong C_2 \times C_2$. On the other hand one can show that non-identity elements of \overline{H}_1 and \overline{H}_2 are all conjugate to

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, \overline{H}_1 and \overline{H}_2 have the same dimension data in $\overline{H} \cong GL(4, \mathbb{F}_2)$.

We show that the groups H_1 and H_2 are non-isomorphic. Since A is a characteristic subgroup of H_1 and H_2 , it is sufficient to show $|Z(H_1/A)| \neq |Z(H_2/A)|$. Obviously $A \triangleleft N \triangleleft H_1, H_2$. Moreover we have

$$\overline{N} = N/A \cong (\mathbb{F}_2)^4$$

as an abelian group and H_1, H_2 act on N/A through the action of $\overline{H}_1, \overline{H}_2$ on $(\mathbb{F}_2)^4$. By Lemma 11.1, \overline{N} is a maximal abelian subgroup of S_n . Therefore,

$$Z(H_i/A) = \overline{N}^{\overline{H}_i},$$

the latter means the fixed point subgroup of the \overline{H}_i action on \overline{N} . By the definition of \overline{H}_1 and \overline{H}_2 , we get

$$\overline{N}^{\overline{H}_1} \cong (\mathbb{F}_2)^2$$

and

$$\overline{N}^{\overline{H}_1} \cong \mathbb{F}_2.$$

Hence, $H_1 \not\cong H_2$. \square

Choose $n = 12$ and a prime $q > 3$. Let $T \subset G$ be the subgroup of diagonal unitary matrices and

$$A = \{\text{diag}\{a_1, a_2, \dots, a_n\} \mid a_1^q = a_2^q = \dots = a_n^q = 1\}.$$

Write

$$N = A \rtimes \left\langle \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \right\rangle$$

and $K = A \rtimes S_n$.

Lemma 11.3. $N_K(N)/N \cong (\mathbb{Z}/12\mathbb{Z})^\times \cong C_2 \times C_2$.

Proof. By Lemma 11.1, $C_G(A) = T$ and $N_G(A) = T \rtimes S_n$. Since $N/A \cong C_{12}$ is a maximal abelian subgroup of S_{12} ,

$$N_K(N)/N \cong N_{S_{12}}(C_{12})/C_{12} \cong (\mathbb{Z}/12\mathbb{Z})^\times.$$

By elementary number theory, $(\mathbb{Z}/12\mathbb{Z})^\times \cong C_2 \times C_2$. \square

Let $H = N_K(N)$ and $\overline{H} = H/N$. Then $\overline{H} \cong C_2 \times C_2$. Write H_1, H_2, H_3, H_4, H_5 for the subgroups of H containing N and with $\overline{H}_1 = H_1/N, \overline{H}_2 = H_2/N, \overline{H}_3 = H_3/N, \overline{H}_4 = H_4/N, \overline{H}_5 = H_5/N$ all the subgroups of \overline{H} , where $|\overline{H}_1| = 1, |\overline{H}_2| = |\overline{H}_3| = |\overline{H}_4| = 2$ and $|\overline{H}_5| = 4$.

Proposition 11.4. *We have*

$$\mathcal{D}_{H_1} + 2\mathcal{D}_{H_5} - (\mathcal{D}_{H_2} + \mathcal{D}_{H_3} + \mathcal{D}_{H_4}) = 0.$$

Proof. Since N is a normal subgroup of H_1, H_2, H_3, H_4 and H_5 , it is sufficient to show the dimension data of the subgroups $\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4, \overline{H}_5$ of \overline{H} have the corresponding linear relation. The latter follows from a consideration on irreducible representations of $C_2 \times C_2$. \square

There exists an example as in Proposition 11.4 when $n = 4$. For $n = 4$, one has $N_G(A)/C_G(A) \cong S_4$. Choose a normal subgroup \overline{N} of S_4 isomorphic to $C_2 \times C_2$ (it is unique) and the subgroups $\overline{H}_1, \overline{H}_2, \overline{H}_3, \overline{H}_4$ of S_4 (in the order of increasing orders) containing \overline{N} . Let H_1, H_2, H_3, H_4 be the corresponding subgroups of $A \rtimes S_4$. One can show that

$$\mathcal{D}_{H_1} + 2\mathcal{D}_{H_4} - (2\mathcal{D}_{H_2} + \mathcal{D}_{H_3}) = 0.$$

Given a closed connected torus T in G , recall that we have a weight lattice $\Lambda_T = \text{Hom}(T, \text{U}(1))$ and a finite group $\Gamma^\circ = N_G(T)/C_G(T)$. Let Φ be the root system of H , Λ be the root lattice of H and ρ_T be the character of the representation of T on \mathbb{C}^n . Write $X = \Lambda_T \otimes_{\mathbb{Z}} \mathbb{R}$. The following lemma is identical to Theorem 4 in [LP]. We state their theorem in a form we needed and sketch the proof.

Lemma 11.5. *Let T be a closed connected torus in G . If there are more than one conjugacy classes of closed connected subgroups H of G with T a maximal torus and acting irreducibly on \mathbb{C}^n , then these representations (H, \mathbb{C}^n) are tensor products of the following list:*

- (1), $n = 2^m$, $H = (\text{Spin}(2m_1 + 1) \times \cdots \times \text{Spin}(2m_s + 1))/Z$ and $\mathbb{C}^n = M_{m_1} \otimes \cdots \otimes M_{m_s}$, where

$$m = m_1 + m_2 + \cdots + m_s,$$

$$Z = \{(\epsilon_1, \dots, \epsilon_s) \mid \epsilon_i = \pm 1, \epsilon_1 \epsilon_2 \cdots \epsilon_s = 1\}$$

and M_m is Spinor representation of $\text{Spin}(2m + 1)$. In this case, $\{\pm 1\}^m \rtimes S_m \subset \Gamma^\circ$.

- (2), $n = 2^{k^2+k} \prod_{1 \leq i \leq k} \frac{i!}{(2i)!} \frac{(m-k-1+2i)!}{(m-k-1+i)!}$, $H = \text{Sp}(m)$ or $\text{SO}(2m)$, $\mathbb{C}^n = V_{(k, k-1, \dots, 1, 0, \dots, 0)}$, $1 \leq k \leq m-1$ and $\frac{k(k+1)}{2}$ is odd; or, $H = \text{Sp}(m)/\langle -I \rangle$ or $\text{SO}(2m)/\langle -I \rangle$, $\mathbb{C}^n = V_{(k, k-1, \dots, 1, 0, \dots, 0)}$, $1 \leq k \leq m-1$ and $\frac{k(k+1)}{2}$ is even. In this case, $\{\pm 1\}^m \rtimes S_m \subset \Gamma^\circ$.
- (3), $n = 27$, $H = G_2$ or $\text{PSU}(3)$, $\mathbb{C}^n = V_\lambda$, $\lambda = 2(e_1 - e_3)$. The weight $\lambda = 2\omega_2$ for G_2 and $\lambda = 2\omega_1 + 2\omega_2$ for A_2 . In this case, $W_{G_2} \subset \Gamma^\circ$.
- (4), $n = 2^{12}$, $H = F_4$, $\text{Sp}(4)/\langle -I \rangle$ or $\text{SO}(8)/\langle -I \rangle$, $\mathbb{C}^n = V_\lambda$, $\lambda = 3e_1 + 2e_2 + e_3$. The weight $\lambda = \omega_3 + \omega_4$ for F_4 , $\lambda = \omega_1 + \omega_2 + \omega_3$ for C_4 and $\lambda = \omega_1 + \omega_2 + \omega_3 + \omega_4$ for D_4 . In this case, $W_{F_4} \subset \Gamma^\circ$.
- (5), Any $n \geq 1$ and irreducible semisimple subgroup H of $U(n)$. In this case, $W_\Phi \subset \Gamma^\circ$, where Φ is the root system of H .

Proof. Since H acts irreducibly on \mathbb{C}^n , the center of H is equal to $H \cap Z(U(n))$. Hence, it is either finite or 1-dimensional. Therefore, the action of W_Φ on X is multiplicity free. As $\Gamma^\circ \supset W_\Phi$, Γ° acts on X multiplicity freely. Let $X = \bigoplus_{1 \leq i \leq m} X_i$ be the decomposition of X into a sum of irreducible summands. Then we have $\Phi = \bigsqcup_{1 \leq i \leq m} \Phi \cap X_i$ and

$$\rho_T = \bigotimes_{1 \leq i \leq m} \rho_i,$$

where ρ_i is the character of a representation of the Lie algebra with root system $\Phi \cap X_i$. Since the character ring $\mathbb{Q}[\Lambda_T]$ is a unique factorization domain and each ρ_i has dominant terms with coefficient 1, the characters $\{\rho_i\}$ are determined by ρ_T . By considering each ρ_i , we may assume that Γ° acts irreducibly and non-trivially on X , which forces H being semisimple. We suppose this since now on.

Since H is assumed to be a closed connected subgroup of $U(n)$, the root lattice Λ is generated by the differences of elements in ρ_T and the integral weight lattice Λ_T is generated by elements in ρ_T . This indicates, both Λ and Λ_T are determined by T . Endowing G with a biinvariant Riemannian metric, this gives Λ a positive definite inner product. We show that Φ° is determined by Λ . This can be proved by induction on the rank l of Λ (which is equal to the rank of Φ°). If $l = 1$, then $\Phi^\circ = A_1$ and it consists of the non-zero elements of Λ of shortest length. Hence, the statement is clear in this case. If $l > 1$, one can show that the shortest non-zero elements of Λ are contained in Φ° . For each element λ , let

$$\lambda = \sum_{1 \leq i \leq m} \alpha_i,$$

$\alpha_i \in \Phi^\circ$ be an expression of this form with m minimal. We have $(\alpha_i, \alpha_j) \geq 0$ for any $1 \leq i \leq j \leq s$ since otherwise $\alpha_i + \alpha_j \in \Phi^\circ$ and we could find an expression with m smaller. Hence, $|\lambda| \geq |\alpha_i|$ for each $1 \leq i \leq m$. Therefore, $m = 1$ and $\lambda \in \Phi^\circ$. Let Φ_1 be the set of shortest non-zero elements of Λ . By the above, Φ_1 is a root system and is contained in Φ° . Let Λ' be the sublattice of elements in Λ orthogonal to elements in Φ_1 and Φ' be the sub-root system of elements in Φ° orthogonal to elements in Φ_1 . Then, Λ' is determined by Λ and Φ' generates Λ' . By induction, Φ' is determined by Λ' . Therefore, Φ° is determined by Λ .

Now we have Φ° determined by Λ . Since Γ° acts irreducibly on X , it acts transitively on the irreducible factors of Φ° . Hence, Φ° is an orthogonal sum of isomorphic irreducible root systems. Write $\Phi^\circ = m\Omega$ where Ω is an irreducible root system. Note that $(B_l)^\circ = lB_1$,

$(C_l)^\circ = D_l$, $(F_4)^\circ = D_4$, $(G_2)^\circ = A_2$ and $\Phi^\circ = \Phi$ if Φ being of type ADE . Hence, if $\Omega \neq A_1$, then $\Phi = \sum_{1 \leq i \leq m} \Phi_i$ with $(\Phi_i)^\circ = \Omega$ for each i . In this case, ρ_T has a decomposition

$$\rho_T = \bigotimes_{1 \leq i \leq m} \rho_i$$

where ρ_i is the character of an irreducible representation of a simple Lie algebra of type Φ_i . Since each ρ_i has dominant terms of coefficient 1, $\{\rho_i \mid 1 \leq i \leq m\}$ are determined by ρ_T . Therefore, we may assume that $m = 1$. That is, we need to consider the ambiguity arising from the pairs $D_l \subset C_l$ ($l \geq 3$), $D_4 \subset F_4$, $C_4 \subset F_4$ and $A_2 \subset G_2$. This follows from some detailed study of the characters of highest weight modules (cf. [LP], Page 395-396). We omit the details here but list the results precisely in the items (2)-(4) of the conclusion of the theorem. If $\Omega \neq A_1$, then

$$\{\pm 1\}^m \subset \Gamma^\circ \subset \{\pm 1\}^m \rtimes S_m.$$

The action of Γ° on Φ° gives a partition of $\{1, 2, \dots, m\}$. This in turn gives a canonical tensor decomposition of ρ_T similar as the above for $\Omega \neq A_1$ case. By this it is enough to consider the case of $\Gamma^\circ = \{\pm 1\}^m \rtimes S_m$. In this case the ambiguity is either as listed in item (1) of the conclusion, or there is no ambiguity as in item (5). \square

Remark 11.6. *In each item (1)-(4), from the above proof, Γ° contains the Weyl group of the largest dimensional Lie group appearing in the ambiguity, which is $\text{Spin}(2m+1)$, $\text{Sp}(m)$, F_4 , G_2 respectively,*

Theorem 11.7. *Let H_1, H_2, \dots, H_s be a list of closed connected subgroups of G acting irreducibly on \mathbb{C}^n . If they are non-conjugate to each other, then their dimension data are linearly independent.*

Proof. By Proposition 3.16, we may assume that H_1, H_2, \dots, H_s have a common maximal torus T . Applying Lemma 11.5, we get a canonical tensor decomposition of ρ_T (the character of the representation of T on \mathbb{C}^n), which in turn gives a decomposition of each H_i . Observe that if each item (1)-(4) occurs, the group Γ° contains a large finite group, i.e., the Weyl group of the largest connected compact Lie group appearing in the ambiguity. By calculation one can show that the dominant weights $2\delta_\Phi$ are non-conjugate to each other under this Weyl group, where $\{\Phi\}$ are the root systems of groups appearing in each item (1)-(4). Therefore, the dominant weights $2\delta_{\Phi_i}$ are non-conjugate to each other under Γ° , where Φ_i is root system of H_i . Hence, the dimension data of H_1, H_2, \dots, H_s are linearly independent. \square

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